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MATHEMATICS 31

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MODULE 8

Exponential and Logarithmic Functions

Distance Learning

Alberta EDUCATION

Mathematics 31

Module 8

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Distance Learning

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Teachers (Mathematics 31)	✓
Administrators	
Parents	
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Other	

Mathematics 31
Student Module Booklet
Module 8
Exponential and Logarithmic Functions
Alberta Distance Learning Centre
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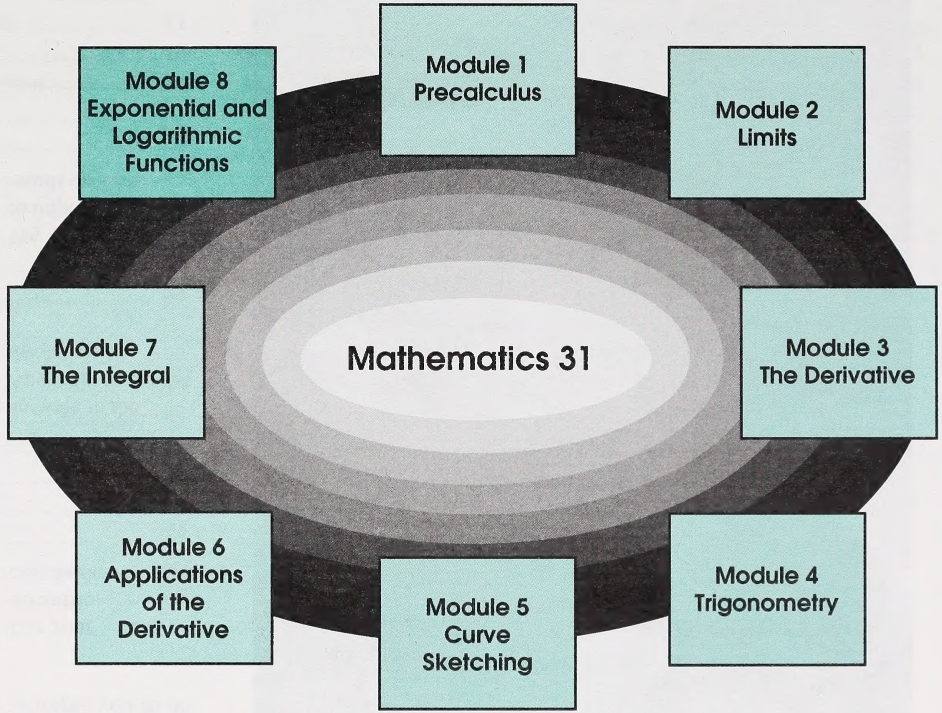
Welcome

WESTFILE INC.

Welcome to Module 8. We hope you'll enjoy your study of Exponential and Logarithmic Functions.

Mathematics 31 contains eight modules. Work through modules in the order given, since several concepts build on each other as you progress in the course.



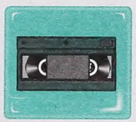
The document you are presently reading is called a Student Module Booklet. You may find visual cues or icons throughout it. Read the following explanations to discover what each icon prompts you to do.

- Use your graphing calculator.

- Use your scientific calculator.

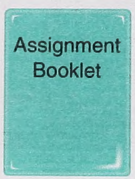
- Use computer software.

- Use the suggested answers in the Appendix to correct the activities.



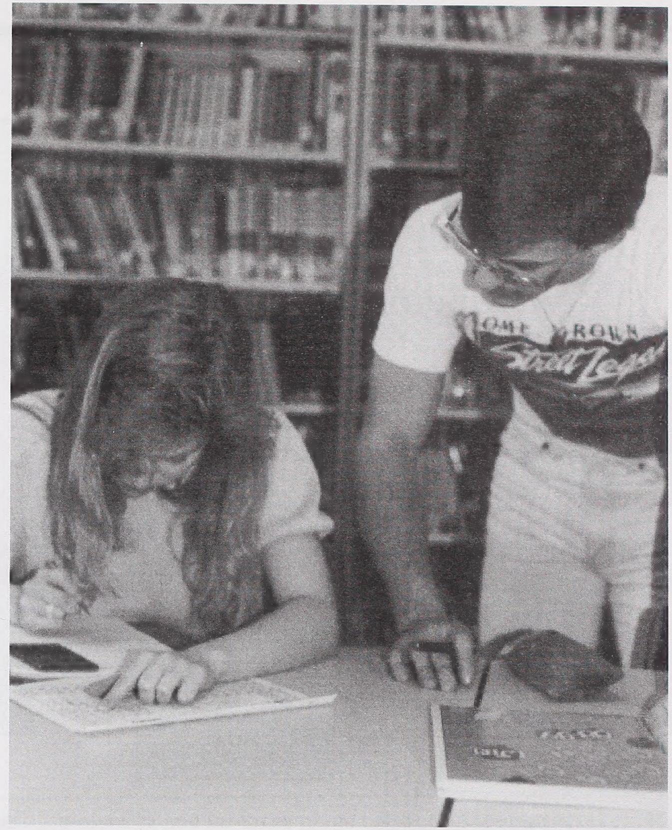
- View a videocassette.

- Pay close attention to important words or ideas.



- Answer the questions in the Assignment Booklet.

Note: Whenever the scientific calculator icon appears, you may use a graphing calculator instead.



There are no response spaces provided in this Student Module Booklet. This means that you will need to use your own paper for your responses. You should keep your response pages in a binder so that you can refer to them when you are reviewing or studying.

CONTENTS

Module Overview

It is human nature to speculate about the future. Mathematical models are often used to make predictions. Population growth, for instance, can be described using exponential and logarithmic functions.

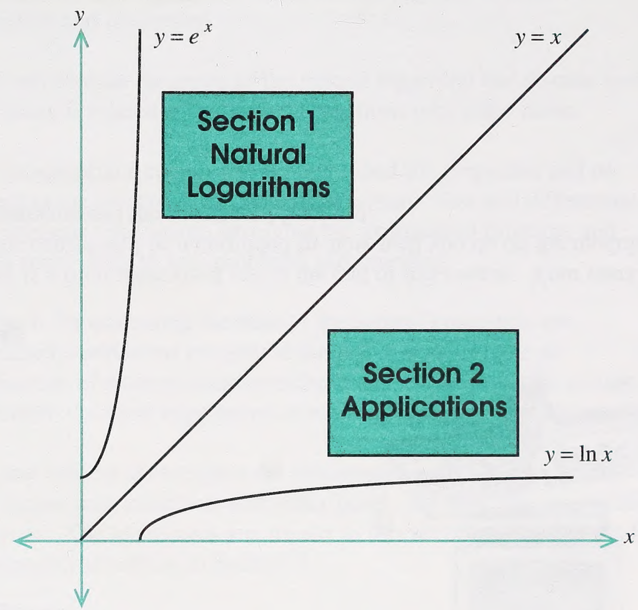
In this module you will investigate exponential and logarithmic functions from both a theoretical and a practical point of view. Recall in Module 7 that you were restricted, when integrating powers, to the form x^n, where $n \neq -1$. Natural logarithms will be defined in Section 1 of this module as a way of bridging that gap.

The exponential function will be defined as the natural logarithm's inverse function. Differentiation and integration techniques, fundamental to this disucssion, occur in each activity. Methods for approximating the base of the natural logarithm, the discussion of limits, and the relationships among logarithms and exponential functions of different bases round out Section 1.

Your study of common and other logarithms in Mathematics 30, and of exponential functions of various bases, will assist you in understanding natural logarithms and the exponential function. Many of the applications from Mathematics 30 will be reintroduced in Section 2. Compound interest, exponential growth and decay, light absorption, and heating and cooling are a few of the everyday situations you will analyse.

It may be helpful to review logarithmic and exponential functions from Mathematics 30 before you begin.

Module 8
Exponential and Logarithmic Functions



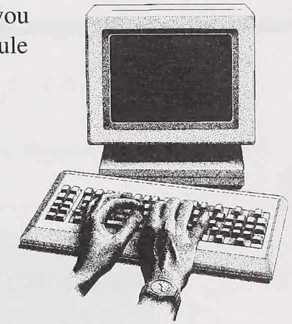
Evaluation

Your mark for this module will be determined by how well you complete the assignments at the end of each section and at the end of the module. In this module you must complete two section assignments and one final module assignment. The mark distribution is as follows:

Section 1 Assignment	**67 marks**
Section 2 Assignment	**15 marks**
Final Module Assignment	**18 marks**
TOTAL	**100 marks**

When doing the assignments, work slowly and carefully. You must do each assignment independently; but if you are having difficulties, you may review the appropriate section in this module booklet.

If you are working on a CML terminal, you will have a module test as well as a module assignment.



Note

There is a final supervised test at the end of this course. Your mark for the course will be determined by how well you do on the module assignments and the supervised final test.

Section 1: Natural Logarithms

Whenever you build a campfire, wait for a cup of coffee to cool, or see heat from Earth's interior brought up to the surface by hot springs and geysers, you can see heat flow from a region of higher temperature to one of lower temperature. This rate of cooling can be described using differential equations. The solutions to those equations involve the integral of the reciprocal function $\int \frac{dx}{x}$.

In this section, you will begin by defining the natural logarithm as an area—the area below the graph of the reciprocal function, $y = \frac{1}{t}$ between 1 and x. As a result, you will be able to evaluate integrals such as $\int \frac{dx}{x}$. The properties of logarithms that you studied in Mathematics 30 are derived for the natural logarithms from its definition and derivative.

You will analyse the graph of the natural logarithm and discuss how that graph is related to logarithmic functions with other bases.

The exponential function will be developed (its properties and its graph) as the inverse of the natural logarithm. You will differentiate and integrate expressions involving the exponential function, and apply those skills to area problems and graphing.

Methods for estimating the base of the natural logarithms are discussed; continuous compound interest is developed as an application of those estimation techniques. Included in this section are limits of simple expressions involving logarithms and exponents.

The last activity investigates the relationships among logarithmic and exponential functions with other bases, and their derivatives and integrals. The techniques you master in this section will be applied to everyday situations in Section 2.

Activity 1: The Natural Logarithm

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Have you noticed that a hot drink cools rapidly just after it is poured, but takes a long time to reach room temperature? According to Newton's Law of Cooling, the rate at which the temperature t of an object changes depends on the temperature difference between it and its surroundings. How long would it take the cup of coffee in the photograph to reach room temperature (20°C)?

From Newton's law,

$$\frac{dT}{dt} \propto (T-20) \text{ or } \frac{dT}{dt} = k(T-20)$$

To find the coffee's temperature T at time t, you must solve the differential equation by separating the variables and integrating both sides.

$$\int \frac{dT}{T-20} = \int k\, dt$$

What is the integral of $\frac{1}{T-20}$? You will have to wait to complete the solution to this problem. In the meantime, you must learn how to integrate a reciprocal function, such as the one on the left side of the equation.

In Module 7, you discovered that $\int \frac{dx}{x}$ could not be evaluated using the rule for powers.

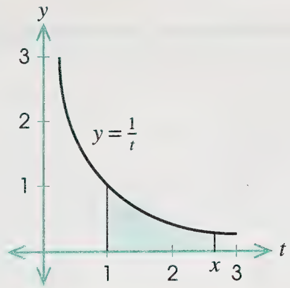
Because the exponent of x in x^{-1} or $\frac{1}{x}$ is $n=-1$, $\int x^n\, dx = \frac{x^{n+1}}{n+1}$ cannot be used. The expression $\frac{x^{-1+1}}{-1+1}$ is not defined.

Therefore, another approach to evaluating the integral of the reciprocal function $\frac{1}{x}$ must be used.

Consider the graph of the reciprocal function $y=\frac{1}{t}$. A new function will be defined as the integral of this reciprocal function. This new function is known as the natural logarithm (abbreviated $\ln x$.)

Now for $x>0$, $\ln x = \int_1^x \frac{1}{t}\, dt$

The symbol **ln *x*** may be read as **lon *x***.



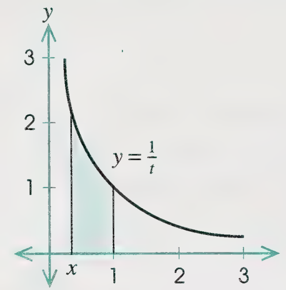
The natural logarithm may be viewed as the area bounded by $y=\frac{1}{t}$, the t-axis, and the vertical lines $t=1$ and $t=x$.

Three conclusions can be drawn immediately.

- If $x>1$, then $\ln x$ will be positive.
- If $x=1$, then $\ln x=0$. The area between $y=\frac{1}{t}$ and the x-axis would be 0 if $x=1$.
- If $0<x<1$, then $\ln x$ is negative. Because x is less than 1,

$$\ln x=\int_1^x \frac{1}{t}\,dt=-\int_x^1 \frac{1}{t}\,dt$$

Look at the following diagram.



You are now ready to evaluate a few natural logarithms.

Example 1

Evaluate $\ln 2$ by doing the following:

Use the graphing software Zap-a-Graph™ and approximate the area between $t=1$ and $t=2$.

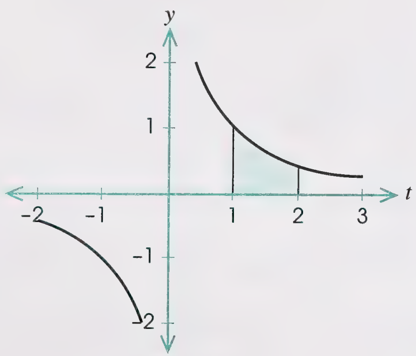
Use a graphing calculator and approximate the area between $t=1$ and $t=2$.

Use a scientific calculator and approximate the area between $t=1$ and $t=2$.

Solution

Zap-A-Graph™ Method

Using Zap-A-Graph™, enter the reciprocal function $y = \frac{1}{x}$. Pull down the Option menu and select "Area". Enter 1 for the lower bound and 2 for the upper bound. You are also given the choice for the number of strips used to subdivide the area.



The greater the number of strips, each strip being approximated by inscribed rectangles, the better the approximation will be. For 1000 strips, the area is approximately 0.693 (to three decimal places).

Graphing Calculator Method

Using a graphing calculator is much more accurate. Even though fewer strips are used, the area of each strip is approximated not by inscribed rectangles, but rather by strips with parabolic tops that lie closely to the given curve. The keystrokes for the CASIO *fx-7700G* graphing calculator are as follows:

Shift | X,θ,T | 1 | ÷ | X,θ,T | Shift | →

1 | Shift | → | 2 | Shift | → | 9 |) | EXE

Lower Bound — Upper Bound — Number of Strips

```
∫(1÷x,1,2,9)
         0.6931471806
```

Note: If you are using a different graphing calculator, then refer to the owner's manual for instructions.

Scientific Calculator Method

Using a scientific calculator (or graphing calculator), the natural logarithm may be found directly using ln.

2 | ln

```
0.693147181
```

Use a graphing calculator to answer question 1.

1. Estimate each logarithm by approximating the area below the graph of $y=\frac{1}{t}$. Check your answer using $\boxed{\ln}$. Sketch the area you are finding.

a. $\ln 5$	**b.** $\ln 4$
c. $\ln 0.25$	**d.** $\ln 2.718\,281\,828\,459$

Check your answers by turning to the Appendix.

Next, you will use the definition of the natural logarithmic function to find derivatives and evaluate integrals.

Remember, integration and differentiation are inverse operations.

Since $\ln x=\int_1^x \frac{1}{t}\,dt$, its derivative is as follows:

$$\begin{aligned}\frac{d}{dx}\ln x &= \frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right]\\ &= \frac{1}{x}\end{aligned}$$

Also, if $y=\ln u$ and $u=f(x)$, then $\frac{dy}{du}=\frac{1}{u}$ and $\frac{du}{dx}=f'(x)$.

Applying the chain rule, $\frac{dy}{dx}=\frac{dy}{du}\bullet\frac{du}{dx}$.

$$\therefore\ \frac{d}{dx}(\ln u)=\frac{1}{u}\bullet\frac{du}{dx}$$

Example 2

Differentiate $y=\ln\left(x^2+1\right)$.

Solution

Apply the chain rule. Since $y=\ln\left(x^2+1\right)$, let $u=x^2+1$.

$$\therefore\ y=\ln u \text{ and } \frac{du}{dx}=2x$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\bullet\frac{du}{dx}\\ &= \frac{1}{u}\bullet\frac{du}{dx}\\ &= \frac{1}{x^2+1}(2x)\\ &= \frac{2x}{x^2+1}\end{aligned}$$

Example 3

Differentiate $y = \ln 2x$.

Solution

Apply the chain rule.

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2x} \bullet \frac{d}{dx}(2x) \\ &= \frac{1(2)}{2x} \\ &= \frac{1}{x}\end{aligned}$$

Example 4

Differentiate $y = (\ln x)^3$.

Solution

Apply the power rule first; then apply the chain rule.

$$\begin{aligned}\frac{dy}{dx} &= 3(\ln x)^2 \frac{d}{dx}(\ln x) \\ &= \frac{3(\ln x)^2}{x}\end{aligned}$$

Example 5

Differentiate $y = x \ln x$.

Solution

Since $y = x \ln x$ is a product, apply the product rule.

$$\begin{aligned}\frac{dy}{dx} &= x \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(x) \\ &= \frac{x}{x} + (\ln x)(1) \\ &= 1 + \ln x\end{aligned}$$

2. Differentiate each function.

a. $y = x^2 \ln x$ **b.** $y = 3\ln x^4$

c. $y = \ln(\sin x)$ **d.** $y = \ln \frac{2x-1}{x-1}$

3. Differentiate the following:

a. $y = x^5 \ln x^5$ **b.** $y = x(\ln x)^2$

c. $y = \ln(\ln x)$ **d.** $y = \ln[\sec(\ln x)]$

Check your answers by turning to the Appendix.

Finally, you will investigate integration. Remember, integration is the inverse of differentiation.

If $\frac{d}{dx}(\ln x) = \frac{1}{x}$, then its integral is as follows:

$$\int \frac{1}{x}\, dx = \ln x + C \qquad (\text{provided } x > 0)$$

Natural logarithms are only defined for positive values of x. To ensure that is the case, the answer should be written as $\ln|x|$ if the positive nature of x is not implied from the context.

In general, $\int \frac{1}{x}\, dx = \ln|x| + C$.

Example 6

Integrate $\int \frac{2}{x+1}\, dx$.

Solution

Assume the primitive is of the form $F(x) = 2a \ln|x+1|$.

$$\begin{aligned} \frac{d}{dx}F(x) &= 2a \bullet \frac{1}{x+1} \bullet \frac{d}{dx}(x+1) \\ &= \frac{2a}{x+1}(1) \\ &= \frac{2a}{x+1} \end{aligned}$$

Solve for a by equating the derivative and integral.

$$\begin{aligned} \frac{2a}{x+1} &= \frac{2}{x+1} \\ a &= 1 \end{aligned}$$

$$\therefore \int \frac{2\, dx}{x+1} = 2\ln|x+1| + C$$

Example 7

Integrate $\int \frac{3x}{x^2+1}\, dx$.

Solution

Use the comparison techniques by assuming the primitive is of the form $F(x) = a \ln\left|x^2+1\right|$.

$$\begin{aligned} \frac{d}{dx}F(x) &= \frac{a}{x^2+1} \bullet \frac{d}{dx}\left(x^2+1\right) \\ &= \frac{2ax}{x^2+1} \end{aligned}$$

Solve for a.

$$\frac{2ax}{x^2+1} = \frac{3x}{x^2+1}$$
$$2a = 3$$
$$a = \frac{3}{2}$$

$$\therefore \int \frac{x}{x^2+1}\,dx = \frac{3}{2}\ln\left|x^2+1\right| + C$$
$$= \frac{3}{2}\ln\left(x^2+1\right) + C$$

Note: Since $x^2+1>0$ for all values of x, absolute value signs are not required.

Example 8

Integrate $\int \tan x\,dx$.

Solution

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

Assume $F(x) = a\ln|\cos x|$.

$$\frac{d}{dx}F(x) = \frac{a}{\cos x}(-\sin x)$$
$$= -a\frac{\sin x}{\cos x}$$

Solve for a.

$$\frac{-a\sin x}{\cos x} = \frac{\sin x}{\cos x}$$
$$a = -1$$

$$\therefore \int \tan x\,dx = -\ln|\cos x| + C$$

4. Integrate the following:

a. $\int \frac{2}{x}\,dx$

b. $\int \frac{4}{x-1}\,dx$

c. $\int \frac{x}{1-x^2}\,dx$

d. $\int \frac{(\ln x)^3}{x}\,dx$

e. $\int \frac{x^2}{x^3+1}\,dx$

f. $\int \frac{\cos x}{2+\sin x}\,dx$

Check your answers by turning to the Appendix.

In this activity, the natural logarithm was defined as the integral of the reciprocal function. The natural logarithm enables you to solve differential equations involving the reciprocal function.

Activity 2: Properties of the Natural Logarithm

In Mathematics 30, you graphed the logarithmic function. You also applied the following properties to simplify expressions and to solve problems, involving common logarithms (base-10 logarithms) and logarithms with other bases.

- $\log_b AB = \log_b A + \log_b B$
- $\log_b \frac{A}{B} = \log_b A - \log_b B$
- $\log_b A^n = n \log_b A$
- $\log_b \sqrt[n]{A} = \frac{1}{n} \log_b A$

Is the graph of the natural logarithm similar? Are its properties similar?

In Activity 1, you were shown that $\ln x = \int_1^x \frac{1}{t}\, dt$ for $x > 0$.

As a consequence, you discovered that $\frac{d}{dx}(\ln x) = \frac{1}{x}$ and

$\int \frac{1}{x}\, dx = \ln|x| + C$.

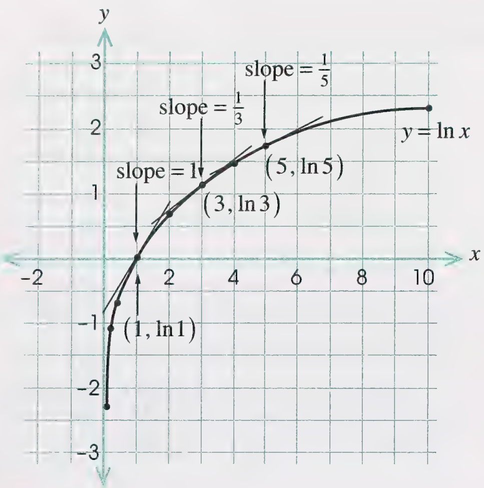
You will now apply these rules to graph $y = \ln x$.

Consider the following table of values. In preparing this table, the functional values have been obtained using a scientific calculator and the values have been rounded to four decimal places for convenience.

Also, since the derivative of $y = \ln x$ is $y' = \frac{1}{x}$, then the third column represents the slope of the graph for each x-value.

x	$\ln x$	$\frac{1}{x}$
$\frac{1}{10}$	-2.3026	10
$\frac{1}{3}$	-1.0986	3
$\frac{1}{2}$	-0.6931	2
1	0	1
2	0.6931	$\frac{1}{2}$
3	1.0986	$\frac{1}{3}$
4	1.3863	$\frac{1}{4}$
5	1.6094	$\frac{1}{5}$
10	2.3026	$\frac{1}{10}$

When the points are plotted, the following graph is obtained. Tangents at various points are shown to illustrate how the slope of the graph changes.

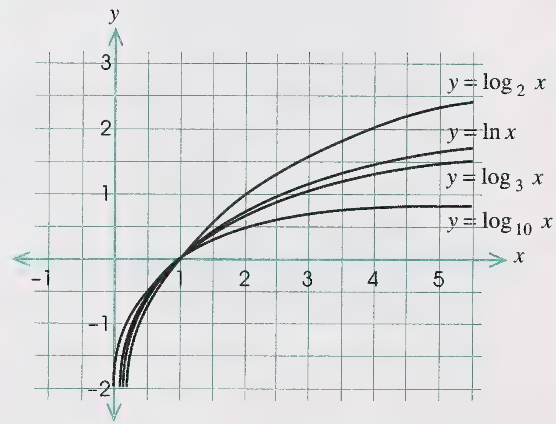


The main features of the graph can be summarized as follows:

- The domain of the function is $x > 0$.
- The range is the set of reals. As $x \to \infty$, $\ln x \to \infty$; as $x \to 0$, $\ln x \to -\infty$.
- Since $\ln x \to -\infty$ as $x \to 0$, the graph is asymptotic to the y-axis.
- Since the derivative of $\ln x$ is $\frac{1}{x}$, where $x > 0$, the derivative is positive throughout the domain of the function. Also, if $\frac{dy}{dx} > 0$, the graph rises to the right.
- Now since $\frac{dy}{dx} = \frac{1}{x}$, then $\frac{d^2 y}{dx^2} = \frac{-1}{x^2}$. Since the second derivative is always negative, the graph is concave downward for all x.
- The x-intercept is 1; thus, there is no y-intercept as the function is defined for only $x > 0$.

If you use a computer graphing program to compare $y = \ln x$, $y = \log_2 x$, $y = \log_3 x$, and $y = \log_{10} x$, you will see that the only difference among them is the steepness of each curve.

Notice that the graph of $y = \ln x$ lies between the curves $y = \log_2 x$ and $y = \log_3 x$.

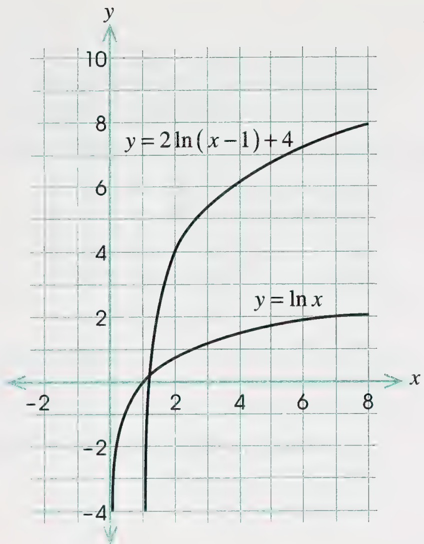


The function $y = \ln x$ may be translated, reflected, and stretched like any other function.

Example 1

Use a graphing calculator (or computer program) to graph $y = 2\ln(x-1)+4$. Describe the graph by comparing it to $y = \ln x$.

Solution



Because $\ln 1 = 0$, the point on the graph of $y = 2\ln(x-1)+4$ that corresponds to $(1, 0)$ on the graph of $y = \ln x$ is $(2, 4)$.

At $x = 2$,
$$\begin{aligned} y &= 2\ln(x-1)+4 \\ &= 2\ln(2-1)+4 \\ &= 2\ln 1+4 \\ &= 2(0)+4 \\ &= 4 \end{aligned}$$

Therefore, the graph of $y = 2\ln(x-1)+4$ is the graph of $y = 2\ln x$ translated horizontally 1 unit and vertically 4 units. In addition, because the leading coefficient is 2, the graph of $y = 2\ln(x-1)+4$ is the graph of $y = \ln x$ stretched vertically by a factor of 2.

Since natural logarithms are not defined for negative values, the domain of $y = 2\ln(x-1)+4$ is $x-1>0$ or $x>1$. Therefore, the curve is asymptotic to $x = 1$.

Use a graphing calculator (or computer program) to answer question 1.

1. Graph each function, and state the domain and the range. Compare each graph with the graph of $y = \ln x$.

a. $y = \ln(2-x)$ **b.** $y = -\ln x + 2$

Check your answers by turning to the Appendix.

As you have seen, the graph of $y = \ln x$ is identical, in form, to the graph of $y = \log_b x$. Do natural logarithms share the same properties with other logarithms? That is, are the following statements true if $x > 0$, $y > 0$, and n is a real number?

- $\ln ax = \ln a + \ln x$
- $\ln \frac{x}{a} = \ln x - \ln a$
- $\ln x^n = n \ln x$
- $\ln \sqrt[n]{x} = \frac{1}{n} \ln x$

The first statement may be verified as follows:

Compare the derivatives of $y = \ln x$ and $y = \ln ax$.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\begin{aligned}\frac{d}{dx}(\ln ax) &= \frac{1}{ax}(a)\\ &= \frac{1}{x}\end{aligned}$$

Since the derivatives of the two functions are equal, then they differ by a constant term at most. That is, $\ln ax = \ln x + C$. To obtain that constant, use the fact that $\ln 1 = 0$. Replace x by 1.

$$\begin{aligned}\ln ax &= \ln x + C\\ \ln a(1) &= \ln 1 + C\\ \ln a &= 0 + C\\ C &= \ln a\end{aligned}$$

$\therefore\ \ln ax = \ln x + \ln a$

2. By applying a procedure similar to the one previously outlined, verify the second and third properties.

a. $\ln \frac{x}{a} = \ln x - \ln a$ **b.** $\ln x^n = n \ln x$

Check your answers by turning to the Appendix.

The fourth property, $\ln \sqrt[n]{x} = \frac{1}{n} \ln x$, follows directly from the third property—the property for powers.

$$\begin{aligned}\ln \sqrt[n]{x} &= \ln x^{\frac{1}{n}}\\ &= \frac{1}{n} \ln x\end{aligned}$$

Now apply these properties.

Example 2

Express $\ln \frac{a^3}{\sqrt{b}}$ in terms of $\ln a$ and $\ln b$.

Solution

$$\begin{aligned}\ln \frac{a^3}{\sqrt{b}} &= \ln a^3 - \ln \sqrt{b}\\ &= 3 \ln a - \frac{1}{2} \ln b\end{aligned}$$

Example 3

Write the following expression as a single natural logarithm.

$$2\ln a - 3\ln b + \frac{1}{3}\ln c - \ln d$$

Solution

$$\begin{aligned}2\ln a - 3\ln b + \frac{1}{3}\ln c - \ln d &= 2\ln a + \frac{1}{3}\ln c - 3\ln b - \ln d\\ &= \left(2\ln a + \frac{1}{3}\ln c\right) - (3\ln b + \ln d)\\ &= \left(\ln a^2 + \ln\sqrt[3]{c}\right) - \left(\ln b^3 + \ln d\right)\\ &= \ln a^2\sqrt[3]{c} - \ln b^3 d\\ &= \ln\frac{a^2\sqrt[3]{c}}{b^3 d}\end{aligned}$$

Apply the rules for natural logarithms in each of the following.

3. Write each of the following as a single natural logarithm. Check your answers using a calculator.

a. $\ln 2 + \ln 3$ **b.** $\ln 12 - \ln 2$

c. $2\ln 4$ **d.** $\frac{1}{2}\ln 49$

4. If $\ln 2 = a$ and $\ln 3 = b$, write each of the following natural logarithms as an expression in a and b.

a. $\ln 6$ **b.** $\ln 12$ **c.** $\ln\sqrt[3]{24}$ **d.** $\ln 1.5$

5. Write each of the following as a single natural logarithm.

a. $4\ln a + \ln b - \ln c$ **b.** $\frac{1}{3}\ln a + \frac{1}{2}\ln b$ **c.** $-\ln a$

6. Simplify $y = \ln(x+1) + \ln(x-1)$.

7. Verify that $\ln\left(x^2 - x\right) = \ln x + \ln(x-1)$.

Check your answers by turning to the Appendix.

Did you know that the natural logarithm is also known as the hyperbolic logarithm or Napierian logarithm?

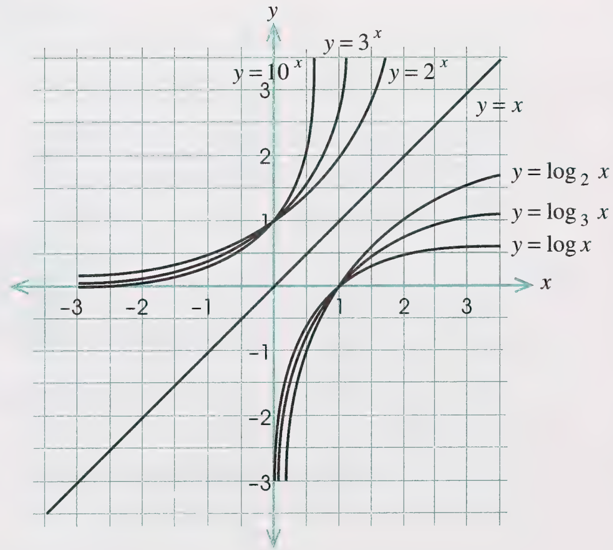
Activity 3: The Exponential Function

When you look in a mirror, everything appears reversed. Also, your image looks like it lies as far behind the mirror as you are in front. In mathematics the graphs of a function and its inverse behave in a similar fashion.

In Mathematics 30, you were shown that logarithmic and exponential functions are inverse functions. Also, recall that the graphs of logarithmic and exponential functions are mirror images in the line $y = x$.

Remember, an **exponential function (base b)** is a function of the form $y = b^x$, where $b > 0$ and $b \neq 1$.

Logarithmic Function	Exponential Function
$y = \log_2 x$	$y = 2^x$
$y = \log_3 x$	$y = 3^x$
$y = \log_{10} x$ (or $\log x$)	$y = 10^x$



In Activity 2, you discovered that the function $y = \ln x$ behaves like logarithmic functions of other bases. Since that is the case, the following questions arise.

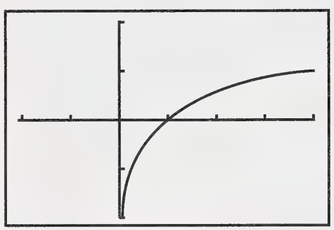
- Is there an exponential function that is the inverse of the natural logarithm?
- What is the base of the exponential function?
- What are the properties of this function and its graph?

Before you can answer these questions, you must return to the graph of $y = \ln x$.

Example 1

Graph $y = \ln x$ on your graphing calculator. Use the Trace feature to approximate the solution to the equation $\ln x = 1$. (Find the value of x for which $y = 1$.) To graph $y = \ln x$ on the CASIO *fx-7700G* graphing calculator, use the following sequence of keystrokes.

Graph ln EXE



Use the Zoom feature to refine your graph.

Use the Trace feature to find an estimate for $\ln x = 1$. For example, you may obtain these values.

- At $x = 2.7182069$, $y = 0.9999724$
- At $x = 2.7183947$, $y = 1.0000415$

These values tell you that the required value of x is between 2.7182 and 2.7183.

1. Interpret the solution to $\ln x = 1$ in terms of the original definition of the natural logarithm, $\ln x = \int_1^x \frac{1}{t}\, dt$.

Check your answer by turning to the Appendix.

The solution to the equation $\ln x = 1$ is represented by the symbol $\boldsymbol{e}$.

Shortly, you will see that e is the **base of the natural logarithm**. Correct to thirteen digits,

$$e = 2.718\,281\,828\,459\ldots$$

Like π, e is an irrational number; it is a non-terminating, non-repeating decimal numeral.

Remember, $\ln e = 1$. Now, what is $\ln e^x$?

x	e^x	$\ln e^x$
-1	e^{-1}	$\ln e^{-1} = -1 \ln e = -1$
0	e^0	$\ln e^0 = 0 \ln e = 0$
1	e^1	$\ln e = 1$
2	e^2	$\ln e^2 = 2 \ln e = 2$
x	e^x	$\ln e^x = x \ln e = x$

Notice that the first and third columns in the table are the same. The natural logarithm $\ln x$, and e^x appear to be inverse operators. One appears to **cancel out** the other. After all, $\ln e^x = x$. Is $y = e^x$ the inverse function of $y = \ln x$?

You will answer that momentarily; first, review the concept of inverse functions.

Suppose $f(x) = 2x$ and $g(x) = \frac{1}{2}x$. If you started with a value of x (say 10) and applied function *f* first, you would obtain $f(10) = 2(10) = 20$. Then, if you applied function *g* to that result, you would get 10 again. $g(20) = \frac{1}{2}(20) = 10$. In other words, when applied consecutively, one function undoes what the other does, as revealed by their composition function.

$$\begin{aligned} g(f(x)) &= g(2x) \\ &= \frac{1}{2}(2x) \\ &= x \end{aligned}$$

These functions are inverse functions. Recall that the inverse of a function can be formed by simply interchanging the values. In the case of function *f*, the following occurs:

- *f*: $y = 2x$
- inverse of *f*: $x = 2y$

The inverse of *f* can be written as $y = \frac{1}{2}x$. This, of course, is function *g*. Now, you can answer the question **Is $y = e^x$ the inverse function of $y = \ln x$?**

The natural logarithm is $y = \ln x$.
The inverse of the natural logarithm is $x = \ln y$.

If $x = \ln e^x$, then $\ln e^x = \ln y$

$$e^x = y$$

Therefore, the inverse of $y = \ln x$ is $y = e^x$. This is called the **exponential function (base *e*)**.

Because $y = \ln x$ and $y = e^x$ are inverse functions, *e* is the base of the natural logarithmic function $y = \ln x$.

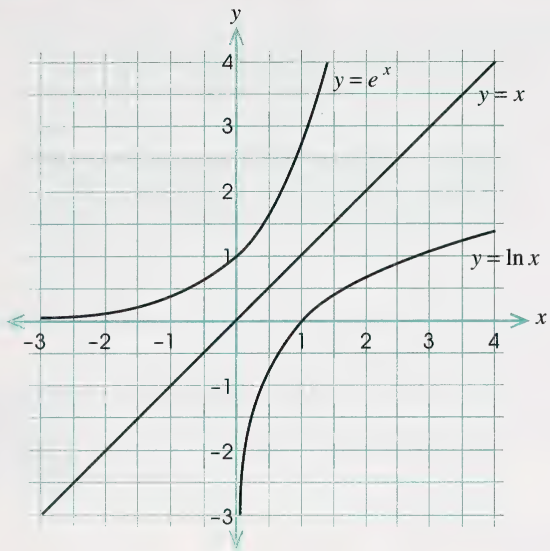
Example 2

Graph $y = e^x$. Compare the graph with $y = \ln x$.

Solution

Construct a table of values.

x	−3	−2	−1	0	1	2	3
e^x	0.050	0.135	0.368	1.000	2.718	7.389	20.086



Since $y = e^x$ is the inverse of $y = \ln x$, the graphs are mirror images in the line $y = x$. Ordered pairs on the graph of $y = \ln x$, such as $(1, 0)$, $(2, 0.693)$, and $(3, 1.099)$, can be used to locate points on $y = e^x$. Simply reverse the coordinates. Therefore, $(0, 1)$, $(0.693, 2)$, and $(1.099, 3)$ lie on $y = e^x$.

Use a graphing calculator (or computer program) to answer the following questions.

2. Draw the graphs of $y = 2^x$, $y = 3^x$, $y = e^x$, and their inverses.

3. State the base of each of the following logarithms:

   a. $y = \ln x$ b. $y = \log x$ c. $y = \log_2 x$

4. State the domain and range of $y = e^x$. How are they related to the domain and range of $y = \ln x$?

5. Locate the intercepts of $y = e^x$.

6. Determine the inverse of $y = \ln(x - 1)$. Draw the graphs of the original function and its inverse.

7. **a.** Graph $y = \ln e^{x}$ and $y = e^{\ln x}$. What do you notice? Explain.

b. What is $e^{\ln 3}$? Use a calculator to confirm your answer.

Check your answers by turning to the Appendix.

Occasionally, you will see $y = e^{x}$ written as $y = \exp(x)$.

Activity 4: Properties of the Exponential Function

In Activity 3, you were shown that the inverse of the natural logarithm $y = \ln x$ is the exponential function $y = e^{x}$. In this activity you will continue the discussion of the exponential function, finding its derivative and antiderivative, describing its graph, and applying your knowledge to areas and maximum and minimum problems.

To find the derivative of $y = e^{x}$, simply express this function as $x = \ln y$ and differentiate implicitly.

$$x = \ln y$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\ln y)$$

$$1 = \frac{1}{y} \bullet \frac{dy}{dx}$$

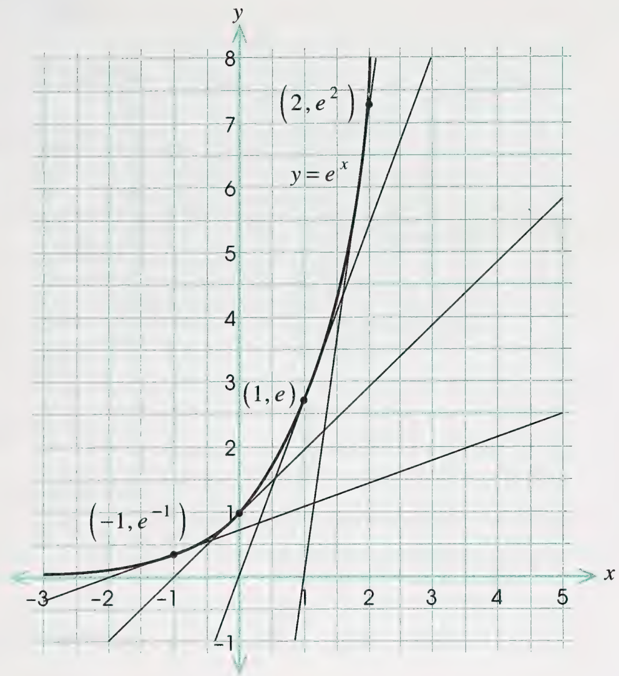
$$y = \frac{dy}{dx}$$

$$\frac{dy}{dx} = e^{x} \text{ since } y = e^{x}$$

$$\therefore \ \frac{d}{dx}\left(e^{x}\right) = e^{x}$$

Here is a function that is virtually indestructable. You can take its derivative over and over again, without changing the function. This, of course, is not true for a polynomial function. Remember, $\frac{d}{dx}\left(x^{n}\right) = nx^{n-1}$.

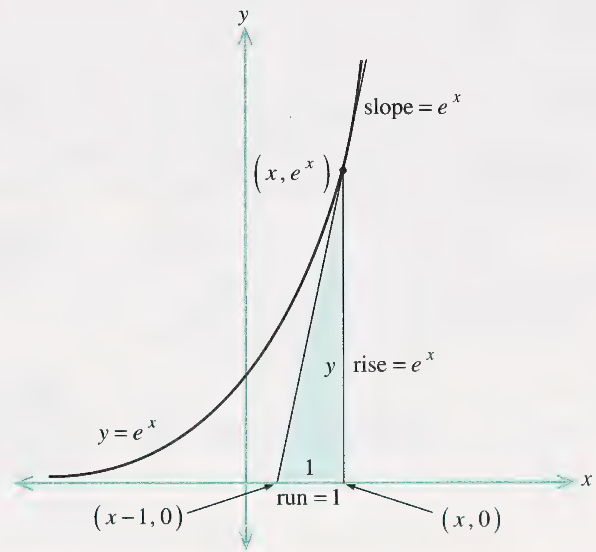
The exponential function has the unusual property that it is its own derivative. How can this be interpreted in terms of its graph?



Remember, the derivative represents the slope of tangent lines. In the preceding diagram, various tangent lines are shown.

Point of Contact	Slope of Tangent Line	x-intercept of Tangent Line
$(2, e^2)$	e^2	1
$(1, e^1)$	e^1	0
$(0, 1)$	1	−1
$(-1, e^{-1})$	e^{-1}	−2

Notice that for each tangent line, the x-intercept is one less than the x-coordinate of the point of contact.



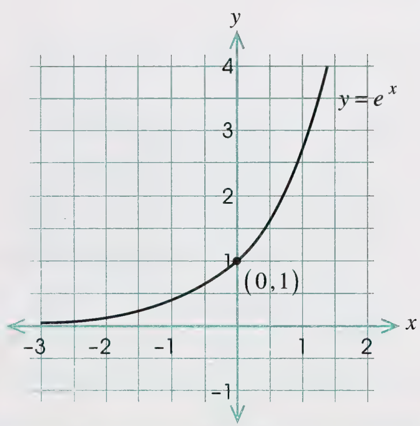
In the general case, if (x, e^x) is the point of contact of a tangent to $y = e^x$, then the rise is the y-coordinate and the run is 1 (as the slope-triangle shows). For the exponential function, its derivative at x is the y-coordinate of the graph for that x-value. For instance, the slope of $y = e^x$ at $x = 4$ is approximately 54.598.

You should now be able to discuss the graph of $y = e^x$.

Example 1

Graph $y = e^x$. State the function's domain, range, intercepts, and asymptotes (if any). Discuss the graph in terms of its first and second derivatives.

Solution



Domain: reals
Range: $\{y \mid y > 0\}$
The x-intercept does not exist.
The y-intercept is 1.

As $x \to \infty$, $e^x \to \infty$; however, as $x \to -\infty$, $e^x \to 0$. The x-axis is a horizontal asymptote.

Since $y' = e^x > 0$, the graph rises to the right throughout its domain. Since $y'' = e^x > 0$, the graph is concave upward for all values of x.

In Example 1, notice that the derivative of $y = e^x$ may be taken any number of times without changing the function. But what is the derivative of $y = e^u$ if $u = f(x)$ and function f is a differentiable function of x?

$$y = e^u$$

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

$$\therefore \ \frac{d}{dx}\left(e^u\right) = e^u \frac{du}{dx}$$

What about antiderivatives?

It follows directly from $\frac{d}{dx}\left(e^{x}\right)=e^{x}$ that $\int e^{x}\,dx=e^{x}+C$.

Example 2

Differentiate each function.

- $y=e^{2x}$
- $y=e^{\sin x}$
- $y=e^{\cos 3x}$

Solution

Apply the formula $\frac{d}{dx}\left(e^{u}\right)=e^{u}\frac{du}{dx}$ in each case.

- $y=e^{2x}$

  Let $u=2x$.

$$\begin{aligned}\therefore\ \frac{d}{dx}\left(e^{u}\right)&=e^{u}\frac{du}{dx}\\&=e^{2x}(2)\\&=2e^{2x}\end{aligned}$$

- $y=e^{\sin x}$

$$\begin{aligned}\frac{dy}{dx}&=e^{\sin x}\frac{d}{dx}(\sin x)\\&=e^{\sin x}\cos x\end{aligned}$$

- $y=e^{\cos 3x}$

$$\begin{aligned}\frac{dy}{dx}&=e^{\cos 3x}\frac{d}{dx}(\cos 3x)\\&=e^{\cos 3x}(-\sin 3x)\frac{d}{dx}(3x)\\&=-3e^{\cos 3x}\sin 3x\end{aligned}$$

1. Differentiate each function.

a. $y=2e^{3x}$ **b.** $y=e^{x}\ln x$ **c.** $y=xe^{x}$

2. Differentiate the following:

a. $y=e^{x^{2}}$ **b.** $y=\sin e^{x}$ **c.** $y=e^{-x}+e^{x}$

Check your answers by turning to the Appendix.

Example 3

Integrate $\int e^{3x}\,dx$.

Solution

Assume the primitive is of the form $F(x) = ae^{3x}$.

Now, $\frac{d}{dx}F(x) = 3ae^{3x}$.

Solve for a.

$$\begin{aligned} 3ae^{3x} &= e^{3x} \\ 3a &= 1 \\ a &= \frac{1}{3} \end{aligned}$$

$$\therefore \int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$$

Example 4

Integrate $\int e^{\cos 3x} \sin 3x\,dx$.

Solution

Assume the primitive is of the form $F(x) = ae^{\cos 3x}$.

$$\frac{d}{dx}F(x) = -3ae^{\cos 3x} \sin 3x$$

Solve for a.

$$-3ae^{\cos 3x} \sin 3x = e^{\cos 3x} \sin 3x$$

$$\begin{aligned} \therefore\ -3a &= 1 \\ a &= -\frac{1}{3} \end{aligned}$$

$$\therefore \int e^{\cos 3x} \sin 3x\,dx = -\frac{1}{3}e^{\cos 3x} + C$$

3. Evaluate each integral.

a. $\int e^{-x}\,dx$

b. $\int \frac{e^{\ln x}}{x}\,dx$

c. $\int 3\sec^2 5xe^{\tan 5x}\,dx$

d. $\int e^{2x} \sin e^{2x}\,dx$

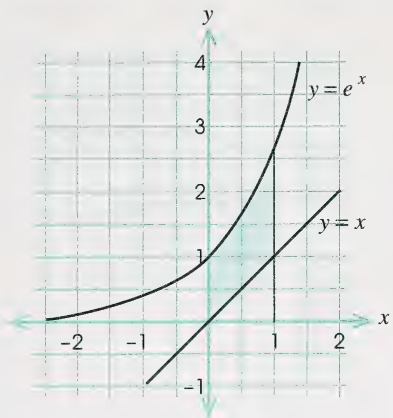
Check your answers by turning to the Appendix.

Now, apply your knowledge to area problems.

Example 5

Find the area between $y = x$ and $y = e^x$, from $x = 0$ to $x = 1$.

Solution



$$
\begin{aligned}
{}_0A_1 &= \int_0^1 [\, f(x) - g(x)] \; dx, \text{ where } f(x) = e^x \text{ and } g(x) = x \\
&= \int_0^1 \left(e^x - x\right) dx \\
&= \left[e^x - \frac{x^2}{2} \right]_0^1 \\
&= \left[e^1 - \frac{1^2}{2} \right] - \left[e^0 - \frac{0^2}{2} \right] \\
&= e - \frac{1}{2} - [1 - 0] \\
&= e - 1.5 \\
&\doteq 1.218\,281
\end{aligned}
$$

4. Find the area between $y = \frac{1}{x}$ and $y = x$, from $x = 1$ to $x = 3$.

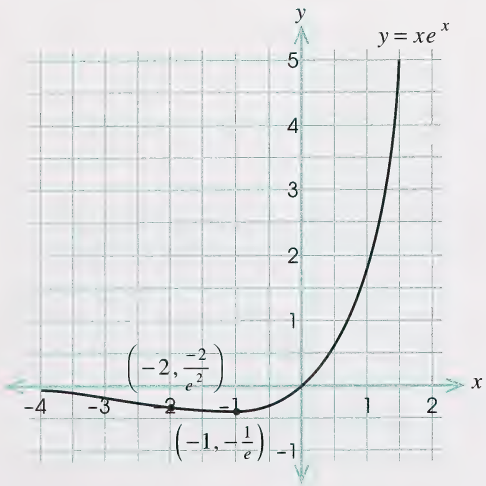
5. Find the area between $y = e^x$ and $y = \frac{1}{x}$, from $x = 1$ to $x = 2$.

Check your answers by turning to the Appendix.

Example 6

Describe the graph of $y = xe^x$.

Solution



- Domain: reals

- Intercepts: When $x = 0$, $y = 0 \bullet e^0 = 0$. Both the *x*-intercept and *y*-intercept are zero.

- Asymptotes: As $x \to -\infty$, $y \to 0$. The horizontal asymptote is the *x*-axis or $y = 0$.

- Intervals of increase or decrease: $y = xe^x$

$$\frac{dy}{dx} = xe^x + e^x(1)$$
$$= e^x(x+1)$$

When $x > -1$, $\frac{dy}{dx} > 0$ and the curve rises to the right.
When $x < -1$, $\frac{dy}{dx} < 0$ and the curve falls to the right.
At $x = -1$, a stationary point occurs.

$$f(-1) = -1\left(e^{-1}\right)$$
$$= -\frac{1}{e}$$
$$\doteq -0.367\,879$$

This is the minimum value of the function.

- Concavity: $\frac{dy}{dx} = e^x(x+1)$

$$\frac{d^2y}{dx^2} = e^x(1) + (x+1)e^x$$
$$= e^x(x+2)$$

The graph is concave upward when $\frac{d^2y}{dx^2} > 0$ or $x > -2$. The graph is concave downward when $\frac{d^2y}{dx^2} < 0$ or $x < -2$.

- Point of inflection: When $\frac{d^2y}{dx^2} = 0$,

$$e^x(x+2) = 0 \text{ or } x = -2$$

$$f(-2) = -2e^{-2} = \frac{-2}{e^2}$$

The point of inflection is $\left(-2, -\frac{2}{e^2}\right)$.

6. Describe each graph fully.

a. $y = x \ln x$ **b.** $y = e^{x^2}$ **c.** $y = \ln x^2$

Check your answers by turning to the Appendix.

You should now be familiar with the exponential function and its properties.

Activity 5: Estimations of *e* and Other Limits

If you began walking at 1 km/h and then doubled your speed over a one-minute interval, you would be walking at 2 km/h. But suppose you increased your speed by 50% every half-minute. How fast would you be walking at the end of one minute?

At the end of the first half-minute you would be walking 1.5 times as fast or at $1(1.5) = 1.5$ km/h. At the end of the next thirty-second interval, you would have increased your speed by 1.5 times again. Therefore, your final speed would be $(1.5)(1.5) = 1.5^2 = 2.25$ km/h.

Suppose you increased your speed by 25% every quarter-minute. What would your speed be at the end of one minute? Remember, your speed would be 1.25 times as fast every quarter-minute.

Time Elapsed (s)	0	15	30	45	60
Speed (km/h)	1	1.25	$(1.25)^2$	$(1.25)^3$	$(1.25)^4$

In this case, your speed would be $(1.25)^4 \doteq 2.441$ km/h. If you increased your speed ten times a minute by 10% $\left(\text{or } \frac{1}{10}\right)$, the expression of your final speed at the end of one minute would be $\left(1+\frac{1}{10}\right)^{10} \doteq 2.594$ km/h. Also, if you increased your speed by $\frac{1}{100}$ a hundred times a minute, your speed would be $\left(1+\frac{1}{100}\right)^{100} \doteq 2.705$ km/h.

Consider what would happen if this process were continued indefinitely (numbers are given to six decimal places):

$$(1+1)^1 = 2$$

$$\left(1+\frac{1}{10}\right)^{10} \doteq 2.593\,742$$

$$\left(1+\frac{1}{100}\right)^{100} \doteq 2.704\,814$$

$$\left(1+\frac{1}{1000}\right)^{1000} \doteq 2.716\,924$$

$$\left(1+\frac{1}{10\,000}\right)^{10\,000} \doteq 2.718\,146$$

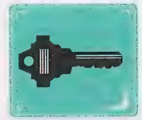
$$\left(1+\frac{1}{100\,000}\right)^{100\,000} \doteq 2.718\,268$$

$$\left(1+\frac{1}{1\,000\,000}\right)^{1\,000\,000} \doteq 2.718\,280$$

$$\left(1+\frac{1}{10\,000\,000}\right)^{10\,000\,000} \doteq 2.718\,282$$

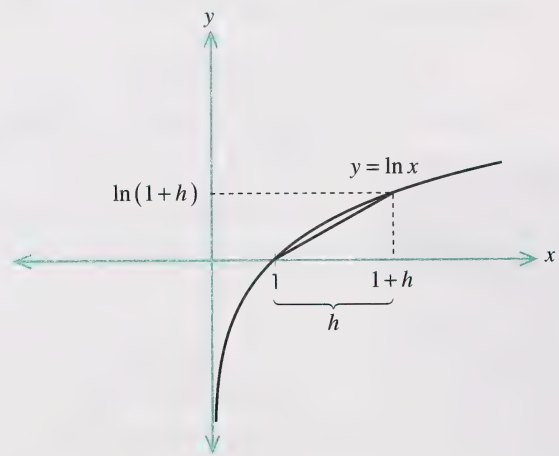
This is almost e.

What is $\left(1+\frac{1}{n}\right)^n$ as $n \to \infty$? Is it $e = 2.718\,281\,828\,459\ldots$? Would you be walking at e km/h?



The following is an explanation for $e = \lim\limits_{n\to\infty}\left[1+\frac{1}{n}\right]^n$.

Replace $\frac{1}{n}$ by h. Therefore, $n = \frac{1}{h}$, and $n \to \infty$ as $h \to 0$. The limit is transformed into $\lim\limits_{h\to 0}\,[1+h]^{\frac{1}{h}}$. This expression suggests the expression for the slope of $f(x) = \ln x$ at $x = 1$.



$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0} \frac{\ln(1+h)-\ln 1}{h}$$
$$= \lim_{h\to 0} \frac{\ln(1+h)-0}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\ln(1+h)$$
$$= \lim_{h\to 0} \ln(1+h)^{\frac{1}{h}}$$

Now, if $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$.

$$\therefore\ 1 = \lim_{h\to 0} \ln(1+h)^{\frac{1}{h}}$$
$$\therefore\ \ln e = \lim_{h\to 0} \ln(1+h)^{\frac{1}{h}}$$
$$\therefore\ e = \lim_{h\to 0} (1+h)^{\frac{1}{h}}$$

$$\therefore\ e = \lim_{n\to\infty}\left[1+\frac{1}{n}\right]^n$$

Recall, $n = \frac{1}{h}$, and $n \to \infty$ as $h \to 0$.

Similarly, it can also be shown that $e^x = \lim_{n\to\infty}\left[1+\frac{x}{n}\right]^n$.

Example 1

Write e^3 as a limit. Use a calculator to approximate that limit. What is the value obtained using the exponent key directly?

Solution

$$e^x = \lim_{n\to\infty}\left[1+\frac{x}{n}\right]^n$$
$$e^3 = \lim_{n\to\infty}\left[1+\frac{3}{n}\right]^n$$

Construct a table for various values of n.

n	$\left[1+\frac{3}{n}\right]^n$
100	19. 218 631 98
1000	19. 995 534 62
10 000	20.076 502 27
100 000	20.084 633 11
1 000 000	20.085 446 54
10 000 000	20.085 527 88

Using the exponential key directly, $e^3 \doteq 20.085\ 537$.

1. Express each power as a limit, and estimate that limit.

a. e^4 **b.** e^{-2}

Check your answers by turning to the Appendix.

One application of the limits formulas for e and e^x is compound interest.

Example 2

You want to invest part of the earnings from your summer job. The number of options the banks offer is baffling.

Suppose you deposited \$1000 into an account bearing interest at 12% per annum. Determine how much you would have after three years if the interest were compounded as follows:

- annually
- semi-annually
- monthly
- daily
- continuously

Solution

In Mathematics 30, you used the formula $A = P(1+i)^n$ to calculate compound interest. Recall that A = accumulated amount, P = principal, i = interest rate per period, and n = number of interest periods.

Compounding Interest Annually

$P = 1000$
$i = 0.12$
$n = 3$

$$\begin{aligned}\therefore\ A &= 1000(1.12)^3 \\ &= 1404.93\end{aligned}$$

The amount after three years is \$1404.93.

Compounding Interest Semi-Annually

$P = 1000$

$i = \frac{0.12}{2}$ (Since the annual rate is 12%, you will receive 6% every six months.)

$= 0.06$

$n = 2(3)$ (The interest is calculated twice a year for three years.)

$= 6$

$\therefore A = 1000(1.06)^6$

$= 1418.52$

The amount after three years is \$1418.52.

Compounding Interest Monthly

$P = 1000$

$i = \frac{0.12}{12}$ (Since the annual rate is 12%, you will receive 1% every month.)

$= 0.01$

$n = 12(3)$ (The interest is calculated twelve times a year for three years.)

$= 36$

$\therefore A = 1000(1.01)^{36}$

$= 1430.76$

The amount after three years is \$1430.76.

Compounding Interest Daily

$P = 1000$

$i = \frac{0.12}{365}$ (Because the annual rate is 12%, you will receive $\frac{12}{365}\%$ every day.)

$n = 365(3)$ (The interest is calculated 365 times a year for three years.)

$$\therefore A = 1000\left(1 + \frac{0.12}{365}\right)^{365(3)}$$

$$= 1433.24$$

The amount after three years is \$1433.24.

Compounding Interest Continuously

If the interest were calculated n times a year, at each interest period you would receive $\frac{12}{n}\%$.

$$\therefore A = 1000\left(1 + \frac{0.12}{n}\right)^{n(3)}$$

If the interest is calculated continuously, $n \to \infty$.

$$\therefore A = 1000\left[\lim_{n\to\infty}\left(1 + \frac{0.12}{n}\right)^{n}\right]^3$$

$$\lim_{n\to\infty}\left(1+\frac{0.12}{n}\right)^n = e^{0.12}$$

$$\therefore\ A = 1000\,e^{0.12(3)}$$
$$= 1433.33$$

The amount after three years is \$1433.33.

Example 2 may be generalized. If the annual rate is r, then the accumulated amount A after t years for a deposit of P dollars, assuming the interest is compounded continuously, is as follows:

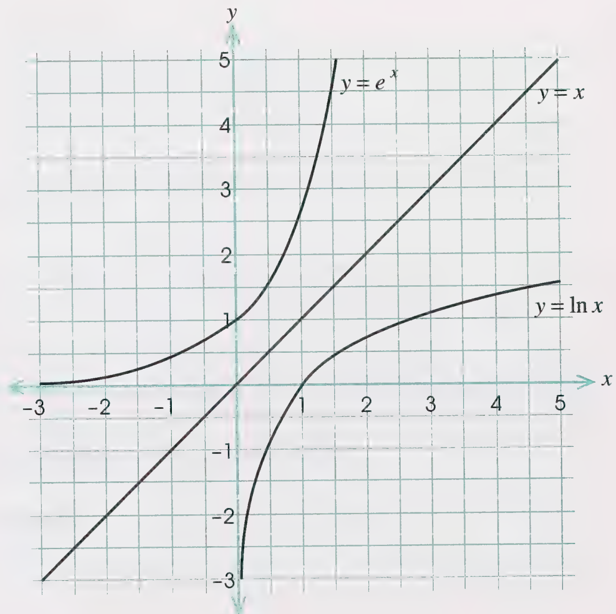
$$A = Pe^{rt}$$

2. How much interest would you earn in four years on a deposit of \$10 000, if the interest rate is 9% per annum, compounded monthly? How much more interest would you receive if the interest were compounded continuously?

3. What interest rate, compounded continuously, will triple your account balance in five years?

Check your answers by turning to the Appendix.

In the latter part of this activity, you will investigate more general limits involving both the exponential and natural logarithmic functions.



From these graphs, it is apparent that $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = \infty$. Also, $\lim_{x\to 0^+} \ln x = -\infty$ and $\lim_{x\to\infty} \ln x = \infty$.

Example 3

Evaluate $\lim\limits_{x \to \pi^-} e^{\sin x}$.

Solution

Replace $\sin x$ by t. If $t = \sin x$, then $t \to 0^+$ as $x \to \pi^-$.

$$\begin{aligned}\lim_{x \to \pi^-} e^{\sin x} &= \lim_{t \to 0^+} e^t \\ &= e^0 \\ &= 1\end{aligned}$$

Example 4

Evaluate $\lim\limits_{x \to \frac{\pi}{2}^+} e^{\tan x}$.

Solution

Replace $\tan x$ by t. If $t = \tan x$, then $t \to -\infty$ as $x \to \frac{\pi}{2}^+$.

$$\begin{aligned}\lim_{x \to \frac{\pi}{2}^-} e^{\tan x} &= \lim_{t \to -\infty} e^t \\ &= 0\end{aligned}$$

Example 5

Evaluate $\lim\limits_{x \to 2^+} \ln(x-2)$.

Solution

Replace $x-2$ by t. If $t = x-2$, then $t \to 0^+$ as $x \to 2^+$.

$$\begin{aligned}\lim_{x \to 2^+} \ln(x-2) &= \lim_{t \to 0^+} \ln t \\ &= -\infty\end{aligned}$$

Example 6

Evaluate $\lim\limits_{x \to \frac{\pi}{2}} \ln|\sin x|$.

Solution

Replace $|\sin x|$ by t. If $t = |\sin x|$, then $t \to 1$ as $x \to \frac{\pi}{2}$.

$$\begin{aligned}\therefore \lim_{x \to \frac{\pi}{2}} \ln|x-2| &= \lim_{t \to 1} \ln t \\ &= \ln 1 \\ &= 0\end{aligned}$$

4. Find each limit.

a. $\lim\limits_{x\to\infty} e^{\frac{3}{x}}$

b. $\lim\limits_{x\to\pi^+} e^{\cos\left(x-\frac{\pi}{2}\right)}$

c. $\lim\limits_{x\to\infty} \ln\left(\dfrac{x}{x+1}\right)$

d. $\lim\limits_{x\to 0^-} \ln\left(e^{\sin x}\right)$

e. $\lim\limits_{x\to -1^+} \ln(x+1)$

Before calculators and computers, mathematical tables were commonly used. The values of logarithms and exponential functions in those tables were obtained using infinite series, for example,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

If you pursue your study of mathematics, you will discover how to derive these formulas.

5. a. Using the preceding infinite series, find $\frac{d}{dx}\left(e^x\right)$.

b. Approximate e^1 using the first six terms of this series. Compare your result with a calculator value.

Check your answers by turning to the Appendix.

Activity 6: Change of Base

A large sheet of paper, 0.01 cm in thickness is folded in half; then it is folded in half a second time, and then a third. If this process is repeated x times, how thick will the folded paper be? If it were possible (and it is not!), how many folds would it take until its thickness were at least the diameter of Earth (approximately 12 800 km)?


NASA

Number of Folds	Thickness (cm)
1	$0.01(2^1)$
2	$0.01(2^2)$
3	$0.01(2^3)$
⋮	⋮
x	$0.01(2^x)$

$1 \text{ km} = 10^5 \text{ cm}; \ 12\,800 \text{ km} = 1.28 \times 10^4 \text{ km}$

$$\therefore \ 0.01\left(2^x\right) \geq \left(1.28 \times 10^4\right)\left(10^5\right)$$
$$0.01\left(2^x\right) \geq 1.28 \times 10^9$$

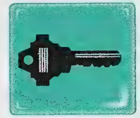
This inequality may be solved by taking natural logarithms of both sides.

$$\begin{aligned} \ln\left[0.01\left(2^x\right)\right] &\geq \ln\left[1.28 \times 10^9\right] \\ \ln 0.01 + x \ln 2 &\geq \ln 1.28 + 9 \ln 10 \\ x &\geq \frac{\ln 1.28 + 9 \ln 10 - \ln 0.01}{\ln 2} \\ &\geq 36.897\,352\ldots \end{aligned}$$

Because there must be a whole-number of folds, $x = 37$. If you could fold the sheet 37 times, it would be thicker than Earth's diameter!

This paper-folding example involved a base-2 exponential function $\left(y = 2^x\right)$ and natural logarithms. This activity investigates the relationships among $y = e^x$, $y = \ln x$, $y = b^x$, $y = \log_b x$, their derivatives, and, their integrals.

First, consider $y = b^x$, where $b > 0$ and $b \neq 1$.



If $y = b^x$, then
$$\begin{aligned} \ln y &= \ln b^x \\ \ln y &= x \ln b \\ y &= e^{x \ln b} \end{aligned}$$

$$\therefore \ b^x = e^{x \ln b}$$

Therefore, a base-b exponential function can be changed into a base-e exponential function.

Example 1

Change 2^x into a power of e.

Solution

$b^x = e^{x \ln b}$,

$2^x = e^{x \ln 2} \qquad (\text{since } b = 2)$

The function $y = 2^x$ is the same as $y = e^{x \ln 2}$.

Use a calculator to answer question 1.

1. Evaluate 2^7 and $e^{7 \ln 2}$. What do you notice? Explain.

2. Convert 4^5 to a power of e.

3. Change $y = 10^x$ to a power of e.

4. Find an equivalent form for $y = e^{x \ln 7}$.

5. Simplify the following:

   a. $e^{\ln 3^2}$ b. $y = e^{\ln 2x}$

Check your answers by turning to the Appendix.

Next, you will look at the derivative and integral of $y = b^x$, where $b > 0$ and $b \neq 0$.

$y = b^x$ is the same as $y = e^{x \ln b}$.

$$\begin{aligned}\therefore\ \frac{d}{dx}\left(b^x\right) &= \frac{d}{dx}\left(e^{x \ln b}\right)\\ &= e^{x \ln b}\,\frac{d}{dx}(x \ln b)\\ &= b^x \ln b\end{aligned}$$

Also, if $u = f(x)$, then by using the chain rule $\frac{d}{dx}\left(b^u\right) = b^u \bullet \frac{du}{dx} \bullet \ln b$.

Finally, since $\frac{d}{dx}\left(b^x\right) = b^x \ln b$, then $\int b^x\, dx = \frac{b^x}{\ln b} + C$.

Example 2

Differentiate $y = 2^x$.

Solution

$$\frac{d}{dx}\left(b^x\right) = b^x \ln b$$

$$\frac{d}{dx}\left(2^x\right) = 2^x \ln 2$$

Example 3

Differentiate $y = 3^{5x+2}$.

Solution

Apply the chain rule.

$$\begin{aligned}\frac{d}{dx}\left(3^{5x+2}\right) &= 3^{5x+2} \bullet \frac{d}{dx}(5x+2) \bullet \ln 3\\ &= 5\left(3^{5x+2}\right)\ln 3\end{aligned}$$

Example 4

Differentiate $y = 10^{\sin 5x}$.

Solution

Apply the chain rule twice.

$$\begin{aligned} \frac{d}{dx}\left(10^{\sin 5x}\right) &= 10^{\sin 5x} \bullet \frac{d}{dx}(\sin 5x) \bullet \ln 10 \\ &= 10^{\sin 5x} \bullet \cos 5x \bullet \frac{d}{dx}(5x) \bullet \ln 10 \\ &= (5 \cos 5x)\left(10^{\sin 5x}\right)(\ln 10) \end{aligned}$$

Example 5

Integrate $\int 3^x \, dx$.

Solution

Apply $\int b^x \, dx = \frac{b^x}{\ln b} + C$.

$$\therefore \int 3^x \, dx = \frac{3^x}{\ln 3} + C$$

Example 6

Integrate $\int \cos x 4^{\sin 2x} \, dx$.

Solution

Use comparison techniques. Assume the primitive is $F(x) = \frac{a}{\ln 4} 4^{\sin 2x}$.

$$\begin{aligned} \frac{d}{dx} F(x) &= \frac{a}{\ln 4} 4^{\sin 2x} (\cos 2x)(2)(\ln 4) \\ &= 2a(\cos 2x) 4^{\sin 2x} \end{aligned}$$

Solve for a.

$$\begin{aligned} 2a(\cos 2x)4^{\sin 2x} &= (\cos 2x)4^{\sin 2x} \\ 2a &= 1 \\ a &= \frac{1}{2} \end{aligned}$$

$$\therefore \int (\cos 2x) 4^{\sin 2x} \, dx = \frac{1}{2 \ln 4} 4^{\sin 2x} + C$$

6. Show that $\frac{d}{dx}\left(b^x\right) = b^x \ln b$ can be used to find the derivative of $y = e^x$.

7. Differentiate the following:

a. $y = 3^x \sin x$ b. $y = 2^{x^2-6}$ c. $y = 2^{\ln x}$

d. $y = 10^{\tan 2x}$ e. $y = \frac{5^x}{x}$

8. Integrate the following:

a. $\int 4^x \, dx$ b. $\int 3(2^x)\, dx$

c. $\int (\sec^2 x) 10^{\tan x}\, dx$

9. Find the area between $y = 2^x$ and the x-axis, from $x = 0$ to $x = 2$.

10. Find the area between $s(x) = 2^x$ and $f(x) = 3^x$, from $x = 0$ to $x = 1.2$.

11. Find the point on $y = 2^x$ with slope 2.

Check your answers by turning to the Appendix.

A calculator allows you to find common (base-10) logarithms and natural logarithms. How would you use a calculator to find logarithms with other bases?

Consider $y = \log_b x$, where $b > 0$ and $b \neq 1$.

In Mathematics 30, you may have written the previous statement as $x = b^y$.

Take the natural logarithm of both sides of the equation.

$$\begin{aligned} \ln x &= \ln b^y \\ \ln x &= y \ln b \\ y &= \frac{\ln x}{\ln b} \\ \therefore \log_b x &= \frac{\ln x}{\ln b} \end{aligned}$$

Example 7

Evaluate $\log_2 8$.

Solution

Method 1: Using Exponential Equations

$$\begin{aligned} y &= \log_2 8 \\ 2^y &= 8 \\ 2^y &= 2^3 \\ y &= 3 \end{aligned}$$

Method 2: Using a Scientific Calculator

$$\log_b x = \frac{\ln x}{\ln b}$$

$$\log_2 8 = \frac{\ln 8}{\ln 2}$$

8 ln ÷ 2 ln =

3.

$\therefore \ \log_2 8 = 3$

Use a calculator to answer question 12.

12. Evaluate each logarithm.

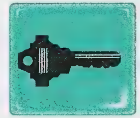
a. $\log_7 6$ **b.** $\log_4 0.2$

Check your answers by turning to the Appendix.

The last part of this activity deals with the derivative of $y = \log_b x$.

$$\log_b x = \frac{\ln x}{\ln b}$$

$$\frac{d}{dx}(\log_b x) = \frac{1}{\ln b} \bullet \frac{d}{dx}(\ln x)$$
$$= \frac{1}{x \ln b}$$



If $u = f(x)$, then applying the chain rule gives the following:

$$\frac{d}{dx}(\log_b u) = \frac{1}{u \ln b} \bullet \frac{du}{dx}$$

Example 8

Differentiate $y = \log_2 x$.

Solution

$$\frac{dy}{dx} = \frac{1}{x \ln 2} \qquad (\text{since } b = 2)$$

Example 9

Differentiate $y = \log_3 (4x - 2)$.

Solution

$$\frac{dy}{dx} = \frac{1}{(4x-2)\ln 3} \bullet \frac{d}{dx}(4x-2)$$
$$= \frac{4}{(4x-2)\ln 3}$$
$$= \frac{2}{(2x-1)(\ln 3)}$$

13. Differentiate the following terms:

a. $y = \log x$ **b.** $y = \log_8 (3x)$

Check your answers by turning to the Appendix.

At the beginning of this activity you investigated base-2 exponential functions through paper folding. There is a practical limit to the number of times a sheet of paper can be folded. What is it?

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

Differentiating functions involving logarithms is often made easier if the orginal functions are simplified first by applying the rules of logarithms.

Example 1

Differentiate $y = \ln \frac{(x-7)^6}{3}$.

Solution

Method 1

Differentiate by applying the chain rule.

$$\begin{aligned} \frac{dy}{dx} &= \frac{3}{(x-7)^6} \bullet \frac{d}{dx} \bullet \frac{(x-7)^6}{3} \\ &= \frac{3(6)(x-7)^5}{(x-7)^6 (3)} \\ &= \frac{6}{x-7} \end{aligned}$$

Method 2

Apply the rules for logarithms; then differentiate.

$$\begin{aligned} y &= \ln \frac{(x-7)^6}{3} \\ &= 6 \ln (x-7) - \ln 3 \\ \frac{dy}{dx} &= \frac{6}{x-7} - 0 \\ &= \frac{6}{x-7} \end{aligned}$$

The advantage of Method 2 is apparent when working with quotients.

Example 2

Differentiate $y = \ln \dfrac{2}{\sqrt{x-2}}$.

Solution

$$\begin{aligned} y &= \ln \frac{2}{\sqrt{x-2}} \\ &= \ln 2 - \frac{1}{2}\ln(x-2) \\ \frac{dy}{dx} &= 0 - \frac{1}{2(x-2)} \\ &= \frac{-1}{2(x-2)} \end{aligned}$$

1. Differentiate each of the following terms.

   a. $y = \ln \dfrac{x-2}{x-3}$

   b. $y = \ln \dfrac{\sqrt{x-1}}{x}$

   c. $y = \ln 3^{x-5}$

   d. $y = \ln 5(x-8)^{11}$

2. Find the derivative of $y = \log 3x$.

Check your answers by turning to the Appendix.

Enrichment

Logarithmic differentiation is a method of differentiation that uses the power of natural logarithms to your advantage. Derivatives of products, quotients, roots, and powers are often simpler to obtain when you take the natural logarithm of both sides of the equation first.

Example 1

Differentiate $y = 3^x$ using logarithmic differentiation.

Solution

Take the natural logarithm of both sides.

$$\ln y = \ln 3^x$$
$$\ln y = x \ln 3$$

Differentiate implicitly, and solve for $\frac{dy}{dx}$.

$$\begin{aligned} \frac{1}{y} \bullet \frac{dy}{dx} &= \ln 3 \\ \frac{dy}{dx} &= y \ln 3 \\ \frac{dy}{dx} &= 3^x \ln 3 \qquad \left(\text{Replace } y \text{ by } 3^x.\right) \end{aligned}$$

Example 2

Differentiate $y=\frac{(x-4)^5}{x+1}$, where $x>4$.

Solution

$$y=\frac{(x-4)^5}{x+1}$$

$$\ln y=\ln\frac{(x-4)^5}{x+1}$$

$$\ln y=5\ln(x-4)-\ln(x+1)$$

$$\frac{1}{y}\bullet\frac{dy}{dx}=\frac{5}{x-4}-\frac{1}{x+1}$$

$$\frac{1}{y}\bullet\frac{dy}{dx}=\frac{5(x+1)-1(x-4)}{(x-4)(x+1)}$$

$$\frac{1}{y}\bullet\frac{dy}{dx}=\frac{5x+5-x+4}{(x-4)(x+1)}$$

$$\frac{1}{y}\bullet\frac{dy}{dx}=\frac{4x+9}{(x-4)(x+1)}$$

$$\frac{dy}{dx}=\frac{y(4x+9)}{(x-4)(x+1)}$$

$$=\frac{(x-4)^5(4x+9)}{(x+1)(x-4)(x+1)}$$

$$=\frac{(x-4)^4(4x+9)}{(x+1)^2}$$

1. Use logarithmic differentiation to find $\frac{dy}{dx}$ of each of the following.

   a. $y=x^{\cos x}$, where $x>0$

   b. $y=x^n$, where $x>0$

   c. $y=x^x$, where $x>0$

   d. $y^2=x(x-1)$, where $x>1$

   e. $y=\frac{(2x-1)^{\frac{1}{3}}(x+1)^{\frac{1}{2}}}{x-1}$, where $x>1$

2. Use implicit differentiation to find the derivative of $y^2=x(x-1)$. Compare your answer to your answer to question 1.d. What do you notice?

Check your answers by turning to the Appendix.

Conclusion

You began this section defining the natural logarithm as an area below the graph of the reciprocal function $y=\frac{1}{t}$, from $t=1$ to $t=x$. Specifically, $\ln x=\int_1^x \frac{1}{t}\,dt$.

In turn, you defined the exponential function as the inverse of the natural logarithm. The derivatives and integrals of these functions, and of logarithmic and exponential functions with other bases, lay a foundation for modelling a variety of real-world situations, such as heating and cooling and compound interest.

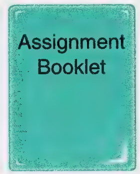
In particular, in this section you should be able to do the following:

- Define exponential and logarithmic functions.
- Recognize that exponential and logarithmic functions are inverses.
- Differentiate logarithmic and exponential functions.
- Analyse the graphs of $y = \ln x$ and $y = e^x$, and approximate the slopes of those graphs for various values of x.
- Use limit theorems to evaluate the limits of simple exponential and logarithmic functions.
- Explain that e may be defined as a limit, and estimate the values of the limits of e and e^x.
- Find the derivatives of logarithmic functions having bases other than e.
- Find the derivatives and antiderivatives of exponential functions having bases other than e.
- Evaluate maxima and minima of given functions involving exponential and logarithmic functions.
- Find areas bounded by exponential, logarithmic, or reciprocal functions.

If you are uncertain of any of these concepts or procedures, review the activites in which they occur.

Now that you have finished this section, pour yourself a hot chocolate. But wait, don't drink it! Place a thermometer in the hot chocolate, and record the temperature at regular intervals as the chocolate cools. Graph the temperature versus time. Notice the curve is not linear but logarithmic. Did you ever think that waiting for a drink to cool was so complicated?

Assignment



You are now ready to complete the section assignment.

Section 2: Applications

Population growth is a global concern. With the increase in human population, wilderness areas and natural habitats have declined. Many animal species are endangered, and people have come to recognize their responsibilites in preserving the natural environment and maintaining the diversity and the populations of other species. Exponential and logarithmic functions are useful tools in modelling human and animal populations, and in making predictions about future numbers.

In Activity 1 of this section, you will examine natural or exponential growth. When the principles of this activity are applied to the population size of people, animals, bacteria, or to the size of investments, a number of questions will be addressed. How fast is it growing? How does size affect growth? What is the equation describing that growth? How can that equation be used to make predictions? Are there different ways of modelling that growth?

In Activity 2, you will investigate natural or exponential decay. Again, the principles may be applied in a number of differing contexts: radioactive decay, thermal cooling, depreciation, the absorption of light, and so on.

As you will see, the calculus of exponential and logarithmic functions has many important applications in modelling natural phenomena, and areas as seemingly remote as biological sciences and economics are interrelated.

Activity 1: Exponential Growth

Two centuries ago, millions of buffalo roamed the prairies of North America. After almost being hunted to extinction in the nineteenth century, remnants of the original herds have made a limited comeback in protected environments such as national parks.

In this activity, you will investigate how the growth of animal and human populations can be modelled using exponential functions.

Your work in this section will involve solving differential equations of the form $\frac{dy}{dt} = ky$, where $y > 0$.

The equation $\frac{dy}{dt} = ky$ may be written $\frac{1}{y} \bullet \frac{dy}{dt} = k$.

$$\therefore \int \frac{dy}{y} = \int k\,dt \text{ or } \ln y = kt + C$$

The constant of integration is normally evaluated from initial conditions.

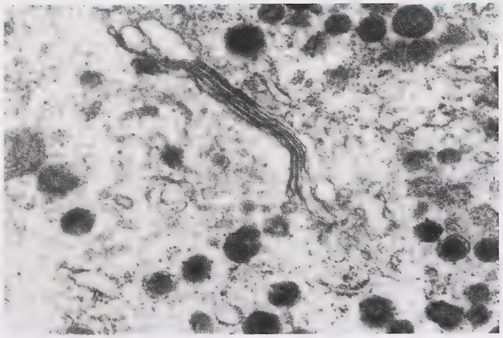
For instance, suppose $y = y_0$ when $t = 0$.

$$\therefore \ln y_0 = k(0) + C$$
$$C = \ln y_0$$

$$\therefore \ln y = kt + \ln y_0$$
$$\ln y - \ln y_0 = kt$$
$$\ln \frac{y}{y_0} = kt$$
$$\frac{y}{y_0} = e^{kt}$$
$$y = y_0 e^{kt}$$

This equation describes **exponential (or natural) growth** provided $k > 0$. You will apply this result in a variety of situations.

Example 1



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The rate of growth $\frac{dy}{dt}$ of a bacterial culture is proportional to the number y of bacteria present at time t (in seconds). If the number of bacteria increases from 1000 to 3000 in 100 s, answer each of the following questions:

- Determine the function $y = f(t)$ that describes the number y of bacteria present at any time t.
- How many bacteria will there be after three minutes?
- How long will it take for the bacteria population to reach 10 000?

Solution

Since $\frac{dy}{dt} \propto y$, $\frac{dy}{dt} = kt$ and $y = y_0 e^{kt}$, where $y = y_0$ at $t = 0$.

At $t = 0$ s, $y = 1000$.

$\therefore\ y = 1000e^{kt}$

To find k, use $y = 3000$ at $t = 100$ s.

$$\begin{aligned} 3000 &= 1000e^{100k} \\ 3 &= e^{100k} \\ \ln 3 &= \ln e^{100k} \\ \ln 3 &= 100k \\ k &= \frac{\ln 3}{100} \end{aligned}$$

$\therefore\ y = 1000e^{(\ln 3)\frac{t}{100}}$

To find the number of bacteria after three minutes (or 180 s), find y when $t = 180$ s.

$$\begin{aligned} y &= 1000e^{(\ln 3)\frac{t}{100}} \\ &= 1000e^{(\ln 3)\frac{180}{100}} \\ &= 1000e^{1.8\ln 3} \\ &\doteq 7225 \qquad \text{(The number of bacteria must be a whole number.)} \end{aligned}$$

There are approximately 7225 bacteria after three minutes.

To find the time needed to increase the bacteria population to 10 000, find t when $y = 10\,000$.

$$\begin{aligned} 10\,000 &= 1000e^{(\ln 3)\frac{t}{100}} \\ 10 &= e^{(\ln 3)\frac{t}{100}} \\ \ln 10 &= \ln e^{(\ln 3)\frac{t}{100}} \\ \ln 10 &= \frac{(\ln 3)t}{100} \\ (\ln 3)t &= 100 \ln 10 \\ t &= \frac{100 \ln 10}{\ln 3} \\ &\doteq 209.6 \end{aligned}$$

The population will reach 10 000 after approximately 209.6 s.

Example 2

WESTFILE INC.

The population of Canada in 1995 was approximately 30 million. If the population increases by 2.5% per year, approximately how many years will it take to double?

Solution

Let y represent Canada's population and t represent time (in years).

Once again, the growth rate is proportional to size. For instance, 2.5% of 30 million is half of 2.5% of 60 million.

Since $\frac{dy}{dt} \propto y$, $\frac{dy}{dt} = kt$ and $y = y_0 e^{kt}$, where $y = y_0$ at $t = 0$.

The 1995 population is 30 million; therefore, $y_0 = 30\,000\,000$.

$$\therefore\ y = 30\,000\,000\, e^{kt}$$

To find k, use the fact that in one year the population will increase by a factor of 1.025; that is, to increase from 100% to $100\% + 2.5\%$.

$$\begin{aligned} 30\,000\,000(1.025) &= 30\,000\,000\, e^{k(1)} \\ 1.025 &= e^{k} \\ \ln 1.025 &= \ln e^{k} \\ \ln 1.025 &= k \end{aligned}$$

$$\therefore\ y = 30\,000\,000\, e^{(\ln 1.025)t}$$

Find the time needed to increase the population to 60 000 000.

$$\begin{aligned} 60\,000\,000 &= 30\,000\,000\, e^{(\ln 1.025)t} \\ 2 &= e^{(\ln .1025)t} \\ \ln 2 &= \ln e^{(\ln 1.025)t} \\ \ln 2 &= (\ln 1.025)t \\ t &= \frac{\ln 2}{\ln 1.025} \\ &\doteq 28 \end{aligned}$$

Canada's population will double in approximately 28 years.

Each of the following questions deals with exponential growth.

1. Apply the formula for compound interest, $A = P(1+i)^n$, to answer Example 2. Do you get the same answer? Why?

2. A population of dust mites doubles in 30 d. Assuming exponential growth, if the present population is 2 million, what will it be in 90 d? (Answer the question by first finding the number y of dust mites as a function of time t (in days) in each form.)

a. $y = y_0 e^{kt}$

b. $y = y_0\, 2^{\frac{t}{d}}$, where d is the doubling time

c. Why are these forms (a. and b.) equivalent?

3. A population of bacteria, growing exponentially, triples its number in sixty minutes. How long does it take for the number of bacteria to double?

4. A population of rabbits doubles its size every three years. If the growth rate $\frac{dy}{dt}$ is proportional to the number of rabbits y, what is the constant of proportionality?

5. A rare stamp increases in value at a rate proportional to its present value y. If its present value is \$200, and it is expected to increase to \$300 in one year, when will it be worth \$1000?

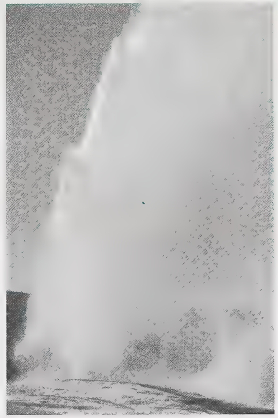
6. A population of red ants is expected to double in 30 d. A nearby population of black ants is expected to double in 45 d. Assuming exponential growth for each population, if there are 30 000 red ants and 45 000 black ants today, when will their populations be the same?

Check your answers by turning to the Appendix.

Animal and human populations cannot grow exponentially indefinitely. What are some of the checks on population growth?

Activity 2: Exponential Decay

Hot springs and geysers are sources of water that have been heated by Earth's interior. These waters cool quickly on a cold day.



In Section 1, you studied that Newton's Law of Cooling states that an object's rate of cooling is proportional to the temperature difference between the object and its surroundings. The greater that difference is, the faster the object cools.

Cooling, radioactive decay, and depreciation may be modelled (as was natural growth) by variations of the proportion $\frac{dy}{dt} \propto y$ or $\frac{dy}{dt} = kt$, where $y > 0$.

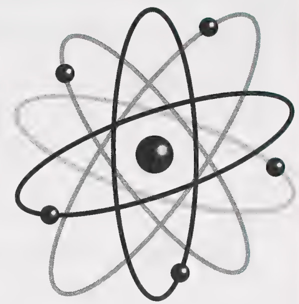
If the derivative $\frac{dy}{dt}$ represents a decreasing function, the constant of proportionality k will be negative. In this case, the differential equation and its solution $y = y_0 e^{kt}$ describe **exponential (or natural) decay**.

Example 1 deals with **radioactive decay**. The rate at which a radioactive isotope decays is proportional to the sample size y; if you double the sample size, you double the number of nuclei disintegrating. This fits the natural decay model $\frac{dy}{dt} \propto y$.

Recall that the time taken for one-half the nuclei in a given sample to disintegrate is called its **half-life**.

Example 1

The rate at which a radioactive isotope disintegrates is proportional to the amount present. If a 30 g sample of Rn^{222} (radon) reduces to only 25 g after one day, what is the half-life of this element? Express your answer to the nearest tenth of a day.



Solution

Let y be the amount of Rn^{222} at time t (in days).

$$\frac{dy}{dt} \propto y \quad \text{or} \quad \frac{dy}{dt} = kt$$

$$\therefore \ y = y_0 e^{kt}, \text{ where } y_0 = 30 \text{ g}$$

$$\therefore \ y = 30 e^{kt}$$

Evaluate k by substituting $y = 25$ when $t = 1$ into the equation.

$$
\begin{aligned}
25 &= 30e^{k(1)} \\
\frac{25}{30} &= e^k \\
\ln\left(\frac{5}{6}\right) &= \ln e^k \\
k &= \ln\left(\frac{5}{6}\right) \\
&\doteq -0.182\,321\,556
\end{aligned}
$$

Remember to expect a negative value.

Note: It is preferrable to use $k = \ln\left(\frac{5}{6}\right)$ in the formula.

$$\therefore\ y = 30e^{\left[\ln\left(\frac{5}{6}\right)\right]t}$$

To determine the half-life, the final sample will contain 15 g.

$$
\begin{aligned}
15 &= 30e^{\left[\ln\left(\frac{5}{6}\right)\right]t} \\
0.5 &= e^{\left[\ln\left(\frac{5}{6}\right)\right]t} \\
\ln 0.5 &= \left[\ln\left(\frac{5}{6}\right)\right]t \\
t &= \frac{\ln 0.5}{\ln\left(\frac{5}{6}\right)} \\
&\doteq 3.8
\end{aligned}
$$

The half-life of Rn^{222} is approximately 3.8 d.

Apply the concepts used in Example 1 in the following.

1. In Mathematics 30, the formula for radioactive decay is given by $y = y_0\, 2^{-\frac{t}{h}}$, where y = amount of the isotope present at time t, y_0 = original amount of the isotope, and h = half-life.

   Translate this equation into the form $y = y_0 e^{kt}$. What is the constant of proportionality k?

2. Carbon dating is based on the assumption that the ratio of the unstable isotope C^{14} to the stable isotope C^{12} in living organisms remains essentially constant. C^{14} is continuously replenished in the organism from the air. When the organism dies, C^{14} is no longer absorbed, and it begins to decay. The half-life of C^{14} is approximately 5700 years.

   A human bone is discovered with only $\frac{1}{100}$ of the C^{14} it would have had when the individual was alive. How long ago did the individual live? Express your answer to the nearest 100 years.

3. The rate at which a radioactive isotope disintegrates is proportional to the amount present. If 80% of a sample of Po^{218} remains after one minute, what percentage would remain after three minutes? Express your answer to the nearest percent.

Check your answers by turning to the Appendix.

Now return to Newton's Law of Cooling. Begin by solving the problem posed in Section 1: Activity 1.

Example 2

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According to Newton's Law of Cooling, an object cools at a rate proportional to the object's temperature difference with its surroundings. A cup of coffee is removed from a microwave, and placed on a counter in a room at 20°C. The coffee initially is 90°C. Five minutes later it is 80°C. What is its temperature at any time t? When will it be 70°C?

Solution

Let the temperature of the coffee be T°C at t min. The temperature difference between the coffee and its surroundings is $(T-20)°\text{C}$. Because the coffee is cooling, this difference is positive; $T-20>0$, or $T>20$. The coffee cools at a rate proportional to the temperature difference.

$$\therefore \quad \frac{dT}{dt} \propto (T-20)$$

$$\frac{dT}{dt} = k(T-20)$$

$$\frac{1}{T-20} \bullet \frac{dT}{dt} = k$$

$$\int \frac{dT}{T-20} = \int k\, dt$$

$$\ln(T-20) = kt + C$$

The initial temperature is 90°C.

$$\therefore \quad \ln(90-20) = k(0) + C$$

$$C = \ln 70$$

$$\therefore \quad \ln(T-20) = kt + \ln 70$$

$$\ln(T-20) - \ln 70 = kt$$

$$\ln\frac{(T-20)}{70} = kt$$

To find k, use the fact that $T = 80°\text{C}$ after $t = 5$ min.

$$\ln\frac{(80-20)}{70} = k(5)$$

$$k = \frac{\ln\left(\frac{6}{7}\right)}{5}$$

$$\therefore \ln\frac{(T-20)}{70}=\frac{\left[\ln\left(\frac{6}{7}\right)\right]t}{5}$$

$$\frac{T-20}{70}=e^{\left[\ln\left(\frac{6}{7}\right)\right]\frac{t}{5}}$$

$$T-20=70e^{\left[\ln\left(\frac{6}{7}\right)\right]\frac{t}{5}}$$

$$T=70e^{\left[\ln\left(\frac{6}{7}\right)\right]\frac{t}{5}}+20$$

Find the time required for the coffee to reach 70°C.

$$70=70e^{\left[\ln\left(\frac{6}{7}\right)\right]\frac{t}{5}}+20$$

$$50=70e^{\left[\ln\left(\frac{6}{7}\right)\right]\frac{t}{5}}$$

$$\frac{5}{7}=e^{\left[\ln\left(\frac{6}{7}\right)\right]\frac{t}{5}}$$

$$\ln\left(\frac{5}{7}\right)=\frac{\left[\ln\left(\frac{6}{7}\right)\right]t}{5}$$

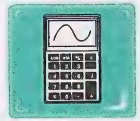
$$t=\frac{5\left[\ln\left(\frac{5}{7}\right)\right]}{\ln\left(\frac{6}{7}\right)}$$

$$\doteq 11$$

The coffee will cool to 70°C in approximately eleven minutes.

The following questions pertain to Newton's Law of Cooling.

4. In Example 2, the temperature difference could have been represented as $D=T-20$. This is because the rate of cooling is proportional to the temperature difference $\frac{dD}{dt}\propto D$. Derive the equation for the temperature difference D at any time t.



Use a graphing calculator (or computer program) to do question 5.

5. Graph the function $T=70e^{\left[\ln\left(\frac{6}{7}\right)\right]\frac{t}{5}}+20$ (from Example 2). Indicate, in terms of the graph, when the coffee reaches 20° C.

6. A roast is taken out of an oven when the meat reaches 85°C. The temperature of the room is 25°C. The meat thermometer reads 80°C five minutes later. When will the roast reach 60°C?

7. The same principle applies for heating as for cooling. That is, the rate of change in temperature of an object being heated is proportional to the temperature difference between it and its surroundings. Some pastries are in a freezer at $-40°$C. They are transferred to an oven warmed to 90°C. If the pastries are 0°C three minutes later, when will they reach 30°C?

Check your answers by turning to the Appendix.

Example 3

The water in a prairie slough is somewhat murky. For every metre below the surface, 40% of the light is absorbed, and only 60% is transmitted. At what depth is 90% of the light absorbed?

Solution

At a depth of 1 m, 60% (or 0.6) of the light penetrating the surface is transmitted. At a depth of 2 m, 60% of that amount or 0.6 of $0.6=(0.6)^2$ is transmitted. At x m, $(0.6)^x$ of the original light is transmitted.

If $y=$ fraction of light transmitted, $y=(0.6)^x$.

If 90% of the light is absorbed, then only 10% is transmitted.

You must solve the equation $0.1=(0.6)^x$ for x.

Take the natural logarithm of both sides.

$$\begin{aligned} \ln 0.1 &= \ln(0.6)^x \\ \ln 0.1 &= x\ln 0.6 \\ x &= \frac{\ln 0.1}{\ln 0.6} \\ &\doteq 4.5 \end{aligned}$$

Ninety percent of the light is absorbed at a depth of about 4.5 m.

Is Example 3 an example of natural decay?

In Section 1: Activity 6, you used the base-b exponential function $b^x=e^{(\ln b)x}$. The function describing light absorption $y=(0.6)^x$ may be written as $y=e^{\left(\ln(0.6)^x\right)}$, which is of the same form as solutions for problems of natural decay: $y=e^{kx}$, where $k<0$. Here, $k=\ln 0.6$. Therefore, light decays exponentially.

8. Depreciation on a car can be described as a loss in value. A car loses value at a rate proportional to its present value. That is, if it were worth twice as much, the rate of loss would be twice as much as well. Suppose you purchase a car for \$20 000. After five years, it is only worth \$3361.40.

a. What is the half-life of your car?

b. What is the annual depreciation, to the nearest percent, of your car? What is the value y of the car expressed as a function of time t (in years)? Give your answer in the form $y=b^x$.

9. A light beam passes through water and is absorbed at the rate of 10% for every metre of depth. What is the half-life of that beam? (That is, at what depth is 50% of the original beam absorbed?)

Check your answers by turning to the Appendix.

A cooling cup of coffee loses heat to its surroundings. Where does the heat go? You may wish to investigate the concept of entropy in thermodynamics.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

You have solved problems in exponential growth using the formula for compound interest $A = P(1+i)^n$, the doubling formula (from Mathematics 30) $y = y_0\, 2^{\frac{t}{d}}$, and the natural growth formula $y = y_0 e^{kt}$.

It is possible to convert among these formulas by applying two concepts.

- An amount doubles when the percentage increase is 100%.
- $2^t = e^{t\ln 2}$

Example

Interpret $y = 1000 \times 2^{\frac{t}{30}}$ if y represents the number of bacteria in a culture at time *t* (in minutes). Modify the interest formula to show the percentage increase in the number of bacteria every minute. Write the original equation using base ***e***.

Solution

The equation $y = 1000 \times 2^{\frac{t}{30}}$ means that the number of bacteria doubles every thirty minutes. Originally, there were 1000 bacteria in the culture.

$$\therefore\ y = 1000 \times 2^{\frac{t}{30}}$$

$$y = 1000\left(2^{\frac{1}{30}}\right)^t$$

$$= 1000(1 + 0.023\,373\,892)^t$$

If you compare this result to $A = P(1+i)^n$, then the percentage increase i is approximately 2.3% every minute.

The original equation $y = 1000 \times 2^{\frac{t}{30}}$ can be written as $y = 1000\, e^{\frac{t(\ln 2)}{30}}$, since $2^t = e^{t\ln 2}$. In that form, natural growth is emphasized.

Use the following information to answer questions 1 to 3.

$y = 3000 \times 2^{\frac{t}{45}}$, where y represents the number of yeast plants in a culture at a time t (in minutes)

1. Explain the meaning of the formula in your own words.
2. Modify the interest formula to show the percentage increase in yeast plants every minute.
3. Write the original equation using base e.

Check your answers by turning to the Appendix.

Enrichment

Exponential growth assumes that populations can grow without bound. This, of course, in a finite world, is not the case! If there are limits on population size, the natural-growth model must be modified. One such modification is the **logistic equation**. If P_{max} is the maximum population a given environment can sustain, then if the current population y is less than that maximum, there is still room $\left(P_{max} - y\right)$ for the population to grow.

The logistic equation is based on the premise that the rate of population growth $\frac{dy}{dt}$ is jointly proportional to both the current population y and the difference from the maximum sustainable population $P_{max} - y$.

$$\frac{dy}{dt} \propto y\left(P_{max} - y\right)$$

$$\therefore \ \frac{dy}{dt} = ky\left(P_{max} - y\right)$$

Note: $\frac{dy}{dt} \to 0$, as $y \to P_{max}$. This means population growth stops when the maximum population is reached.

To solve the differential equation, separate the variables and integrate.

$$\frac{1}{y\left(P_{max} - y\right)} \bullet \frac{dy}{dt} = k$$

$$\int \frac{dy}{y\left(P_{max} - y\right)} = \int k\, dt$$

Before the integral on the left side can be evaluated, the fraction $\frac{1}{y\left(P_{max} - y\right)}$ must be separated into two fractions of the form $\frac{A}{y} + \frac{B}{P_{max} - y}$. This way, each may be integrated using the techniques outlined for reciprocal functions. To find A and B, add; then compare the sum with the original fraction.

$$\begin{aligned}\frac{A}{y}+\frac{B}{P_{\max}-y} &= \frac{A\left(P_{\max}-y\right)+By}{y\left(P_{\max}-y\right)}\\ &= \frac{AP_{\max}-Ay+By}{y\left(P_{\max}-y\right)}\\ &= \frac{AP_{\max}+\left(-Ay+By\right)}{y\left(P_{\max}-y\right)}\\ &= \frac{AP_{\max}+\left(-A+B\right)y}{y\left(P_{\max}-y\right)}\end{aligned}$$

Equate this result to $\frac{1}{y\left(P_{\max}-y\right)}=\frac{1+0y}{y\left(P_{\max}-y\right)}$.

$$\begin{aligned}\frac{1+0y}{y\left(P_{\max}-y\right)} &= \frac{AP_{\max}+\left(-A+B\right)y}{y\left(P_{\max}-y\right)}\\ AP_{\max} &= 1 \text{ and } -A+B=0\\ A &= \frac{1}{P_{\max}} \text{ and } B=A\\ \therefore \frac{1}{y\left(P_{\max}-y\right)} &= \frac{A}{y}+\frac{B}{P_{\max}-y}\\ &= \frac{1}{P_{\max}y}+\frac{1}{P_{\max}\left(P_{\max}-y\right)}\end{aligned}$$

Now, return to the original problem.

$$\int\frac{dy}{y\left(P_{\max}-y\right)}=\int k\,dt$$

$$\int\frac{dy}{P_{\max}y}+\int\frac{dy}{P_{\max}\left(P_{\max}-y\right)}=\int k\,dt$$

Since the current population and $P_{\max}-y$ are positive,

$$\begin{aligned}\therefore \frac{1}{P_{\max}}\ln y-\frac{1}{P_{\max}}\ln\left(P_{\max}-y\right) &= kt+C\\ \ln y-\ln\left(P_{\max}-y\right) &= kP_{\max}t+CP_{\max}\\ \ln\frac{y}{P_{\max}-y} &= kP_{\max}t+CP_{\max}\end{aligned}$$

Evaluate $CP_{\max}$ using the initial population y_0 at $t=0$.

$$\begin{aligned}\ln\frac{y_0}{P_{\max}-y_0} &= kP_{\max}(0)+CP_{\max}\\ CP_{\max} &= \ln\frac{y_0}{P_{\max}-y_0}\end{aligned}$$

$$\therefore \ \ln\frac{y}{P_{\max}-y}=kP_{\max}t+\ln\frac{y_0}{P_{\max}-y_0}$$

$$\ln\frac{y}{P_{\max}-y}-\ln\frac{y_0}{P_{\max}-y_0}=kP_{\max}t$$

$$\ln\left(\frac{y}{P_{\max}-y}\right)\left(\frac{P_{\max}-y_0}{y_0}\right)=kP_{\max}t$$

$$\left(\frac{y}{P_{\max}-y}\right)\left(\frac{P_{\max}-y_0}{y_0}\right)=e^{kP_{\max}t}$$

$$\left(\frac{y}{P_{\max}-y}\right)=\left(\frac{y_0}{P_{\max}-y_0}\right)e^{kP_{\max}t}$$

$$y=\left(P_{\max}-y\right)\left(\frac{y_0}{P_{\max}-y_0}\right)e^{kP_{\max}t}$$

$$y=P_{\max}\left(\frac{y_0}{P_{\max}-y_0}\right)e^{kP_{\max}t}-y\left(\frac{y_0}{P_{\max}-y_0}\right)e^{kP_{\max}t}$$

$$y+y\left(\frac{y_0}{P_{\max}-y_0}\right)e^{kP_{\max}t}=P_{\max}\left(\frac{y_0}{P_{\max}-y_0}\right)e^{kP_{\max}t}$$

$$y\left[1+\left(\frac{y_0}{P_{\max}-y_0}\right)e^{kP_{\max}t}\right]=P_{\max}\left(\frac{y_0}{P_{\max}-y_0}\right)e^{kP_{\max}t}$$

$$y=\frac{P_{\max}\left(\frac{y_0}{P_{\max}-y_0}\right)e^{kP_{\max}t}}{\left[1+\left(\frac{y_0}{P_{\max}-y_0}\right)e^{kP_{\max}t}\right]}$$

$$y=\frac{\left[P_{\max}\left(\frac{y_0}{P_{\max}-y_0}\right)e^{kP_{\max}t}\right]\frac{\left(P_{\max}-y_0\right)}{1}}{\left[1+\left(\frac{y_0}{P_{\max}-y_0}\right)e^{kP_{\max}t}\right]\frac{\left(P_{\max}-y_0\right)}{1}}$$

$$y=\frac{P_{\max}y_0e^{kP_{\max}t}}{P_{\max}-y_0+y_0e^{kP_{\max}t}}$$

$$y=\frac{P_{\max}y_0e^{kP_{\max}t}}{\left(P_{\max}-y_0+y_0e^{kP_{\max}t}\right)}\bullet\frac{e^{-kP_{\max}t}}{e^{-kP_{\max}t}}$$

$$\therefore \ y=\frac{P_{\max}y_0}{y_0+\left(P_{\max}-y_0\right)e^{-kP_{\max}t}}$$

In the following example, you will see how the logistic formula may be applied.

Example

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Biologists estimate that the maximum population of grouse the environment can sustain in an isolated wilderness area in Northern Alberta is 10 000 birds. One year ago, the grouse population in that area was estimated at 4000. Today, it stands at 4500. Assume that the grouse population can be modelled by the logistic equation. Determine the logistic equation for the size of the grouse population y after time t (in years). When will the grouse population reach 7000?

Solution

The logistic equation is as follows:

$$y = \frac{P_{max} y_0}{y_0 + (P_{max} - y_0) e^{-kP_{max} t}}$$

$P_{max} = 10\,000$ and $y_0 = 4000$ at $t = 0$ (that is, time is measured from one year ago). Also, $y = 4500$ at $t = 1$. Determine k.

$$\begin{aligned}
4500 &= \frac{4000(10\,000)}{4000 + (10\,000 - 4000)e^{-k(10\,000)(1)}} \\
4500 &= \frac{4000(10\,000)}{1000\left(4 + 6e^{-k(10\,000)(1)}\right)} \\
4500 &= \frac{4(10\,000)}{4 + 6e^{-k(10\,000)(1)}} \\
45 &= \frac{2(100)}{2 + 3e^{-k(10\,000)(1)}} \\
90 + 135e^{-k(10\,000)(1)} &= 200 \\
e^{-k(10\,000)(1)} &= \frac{110}{135} \\
e^{10\,000k} &= \frac{135}{110} \\
10\,000k &= \ln\left(\frac{27}{22}\right) \\
k &= \frac{\ln\left(\frac{27}{22}\right)}{10\,000}
\end{aligned}$$

Now substitute the known values into the logistic formula.

$$y = \frac{10\,000(4000)}{4000 + (10\,000 - 4000)e^{-\frac{\ln\left(\frac{27}{22}\right)}{10\,000}(10\,000)t}}$$

$$= \frac{10\,000(4000)}{4000 + 6000e^{-\left[\ln\left(\frac{27}{22}\right)\right]t}}$$

$$= \frac{20\,000}{2 + 3e^{-\left[\ln\left(\frac{27}{22}\right)\right]t}}$$

To find the number of years for the population to reach 7000, solve the equation.

$$7000 = \frac{20\,000}{2 + 3e^{-\left[\ln\left(\frac{27}{22}\right)\right]t}}$$

$$7\left\{2 + 3e^{-\left[\ln\left(\frac{27}{22}\right)\right]t}\right\} = 20$$

$$14 + 21e^{-\left[\ln\left(\frac{27}{22}\right)\right]t} = 20$$

$$e^{-\left[\ln\left(\frac{27}{22}\right)\right]t} = \frac{2}{7}$$

$$-\left[\ln\left(\frac{27}{22}\right)\right]t = \ln\left(\frac{2}{7}\right)$$

$$t = \frac{\ln\left(\frac{2}{7}\right)}{-\left[\ln\left(\frac{27}{22}\right)\right]}$$

$$\doteq 6.1$$

The grouse population will reach 7000 after approximately 6.1 years (or 5.1 years from now).

Use a graphing calculator (or computer program) to answer question 1.

1. Sketch the logistic equation from the preceding example. What do you notice about the graph?

2. Thirty wolves were released into a national park that can support a maximum population of 200 wolves. Two years later, there were 40 wolves in the park. Assuming that the wolf population satisfies the logistic equation, when will the wolf population reach 100?

Check your answers by turning to the Appendix.

Conclusion

As you have seen, calculus of exponential and logarithmic functions has many important applications in modelling natural phenomena, particularly in the areas of growth and decay. Compound interest, depreciation, population growth and decline, radioactive decay, and heating and cooling may be described using similar differential equations.

You should be able to do the following:

- Relate natural growth and decay to the differential equations $y' = ky$ or $y' = k(y - y_0)$.
- Solve natural growth and decay problems starting from the differential equations $y' = ky$ or $y' = k(y - y_0)$.
- Fit exponential models to observed data.

If you had difficulty applying any of these concepts, review the activities.

Mathematical models for human or animal population growth are approximations at best. These models may fit the data in the short term, but cannot be used, with any accuracy, to predict populations far into the future. There are simply too many factors that affect population size. You may wish to investigate some of the population models that arise out of the branch of mathematics called Chaos Theory.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Module Summary

In this module you investigated logarithmic and exponential functions. The natural logarithm $\ln x$ was defined in Section 1 as an area between the graph of $y = \frac{1}{t}$ and the t-axis. As a result, you were able to integrate the reciprocal function, completing your discussion of the power rule begun in Module 7.

The exponential function $y = e^x$ was defined as the natural logarithm's inverse function. The nature of that relationship is fundamental to your understanding of e and its evaluation.

In Section 2 you pursued applications of exponents and logarithms—compound interest, the exponential growth of populations, radioactive decay, light absorption, and heating and cooling. You should now be able to model these and other similar problem-solving situations, and discuss them using that model.

Do all insect and animal populations grow exponentially? What prevents the butterfly population in the photograph from growing indefinitely?

This completes Mathematics 31. Good luck in your continued study of mathematics.

Final Module Assignment

Assignment Booklet

You are now ready to complete the final module assignment.

COURSE SURVEY FOR MATHEMATICS 31

Please evaluate this course and return this survey with your final module assignment. This course is designed in a new distance learning format, so we are interested in your responses. Your constructive comments will be greatly appreciated, as future course revisions can then incorporate any necessary improvements.

Name ______________________________ Age ☐ under 19 ☐ 19 to 40 ☐ over 40

Address ______________________________ File No. ______________________________

______________________________ Date ______________________________

____

Design

1. This course contains a series of student module booklets. Do you like the idea of separate booklets?

____

____

2. Have you ever enrolled in a correspondence or distance learning course that arrived as one large volume?

   ☐ Yes ☐ No If yes, which style do you prefer?

____

____

3. The student module booklets contain a variety of self-assessed activities. Did you find it helpful to be able to check your work and have immediate feedback?

   ☐ Yes ☐ No If yes, explain.

____

____

4. Were the questions and directions easy to understand?

   ☐ Yes ☐ No If no, explain.

____

____

5. Each section contains follow-up activities. Which type of follow-up activity did you choose?

- ❑ mainly extra help
- ❑ mainly enrichment
- ❑ a variety
- ❑ none

Did you find these activities beneficial?

❑ Yes ❑ No If no, explain.

____

____

6. Did you understand what was expected in the assignment booklets?

❑ Yes ❑ No If no, explain.

____

____

7. The course materials were designed to be completed by students working independently at a distance. Were you always aware of what you had to do?

❑ Yes ❑ No If no, provide details.

____

____

8. This course may include drawings, photographs, and charts.

a. Did you find it helpful to have these visuals?

❑ Yes ❑ No Comment on the lines below.

____

____

b. Did you find the variety of visuals motivating?

❑ Yes ❑ No Comment on the lines below.

____

____

9. Suggestions for audiocassette and videocassette activities may have been included in the course. Did you make use of these media options?

   ❑ Yes ❑ No Comment on the lines below.

   ____

   ____

Only students enrolled in a Junior High course need to complete the following question.

10. Students are often directed to work with their learning facilitator. How well did you work as a team?

    Student's comments: ____

    ____

    ____

    ____

    Learning Facilitator's comments: __________________________________

    ____

    ____

    ____

Course Content

1. Was enough detailed information provided to help you learn the expected skills and ojectives?

   ❑ Yes ❑ No Comment on the lines below.

   ____

   ____

2. Did you find the work load reasonable?

   ❑ Yes ❑ No If no, explain.

   ____

   ____

3. Did you have any difficulty with the reading level?

❑ Yes ❑ No Please comment.

4. How would you assess your general reading level?

❑ poor reader ❑ average reader ❑ good reader

5. Was the material presented clearly and with sufficient depth?

❑ Yes ❑ No If no, explain.

General

1. What did you like least about the course?

2. What did you like most about the course?

Additional Comments

Only students enrolled with the Alberta Distance Learning Centre need to complete the remaining questions.

1. Did you contact the Alberta Distance Learning Centre for help or information while doing your course?

   ❑ Yes ❑ No If yes, approximately how many times? ______

   Did you find the staff helpful?

   ❑ Yes ❑ No If no, explain.

   ____

   ____

2. Were you able to fax any of your assignment response pages?

   ❑ Yes ❑ No If yes, comment on the value of being able to do this.

   ____

   ____

3. If you mailed your assignment response pages, how long did it take for their return?

   ____

   ____

4. Was the feedback you received from your correspondence or distance learning teacher helpful?

   ❑ Yes ❑ No Please comment.

   ____

   ____

   ____

   ____

Thanks for taking the time to complete this survey. Your feedback is important to us. Please return this survey with your final module assignment.

Instructional Design and Development
Alberta Distance Learning Centre
Box 4000
Barrhead, Alberta
T7N 1P4

APPENDIX

Glossary

Suggested Answers

Glossary

Common logarithm: a base-10 logarithm

e: the base of the natural logarithm; $e \doteq 2.718\ 281\ 828$

Exponential function (base b): a function of the form $y = b^x$ where $b > 0$ and $b \neq 1$

Exponential function (base e): the inverse of the natural logarithmic function; $y = e^x$

Exponential (or natural) decay: decline modelled by $y' = ky$, where $k < 0$

Exponential (or natural) growth: growth modelled by $y' = ky$, where $k > 0$

Half-life: time for one-half the nuclei in a radioactive isotope to decay

Logarithm (base b): a logarithm written $y = \log_b x$, satisfying the equation $x = b^y$, where $b > 0$ and $b \neq 1$

Natural logarithm: the base-e logarithm; $\ln x$ being defined as the area between $y = \frac{1}{t}$ and the t-axis, from $t = 1$ to $t = x$

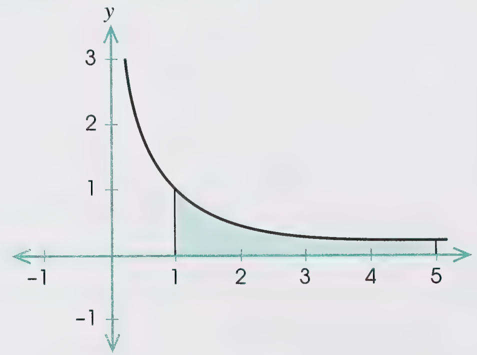
Newton's Law of Cooling: cooling of an object at a rate proportional to the temperature difference between the object and its surroundings

Radioactive decay: disintegration of the nucleus of an isotope, the rate of which may be described by natural decay

Suggested Answers

Section 1: Activity 1

1. a.



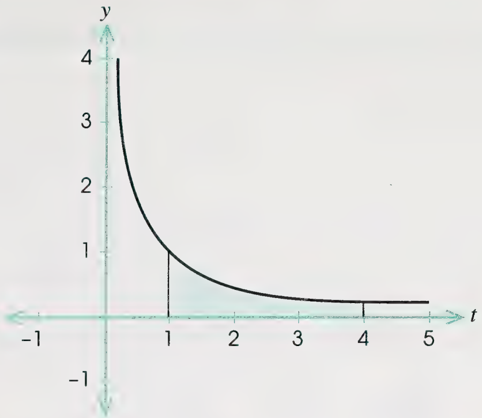
Shift X,θ,T 1 ÷ X,θ,T Shift → 1 Shift → 5 Shift → 9) EXE

The area is approximately 1.609 437 91.

Check

$\ln 5 = 1.609\ 437\ 912$

b.



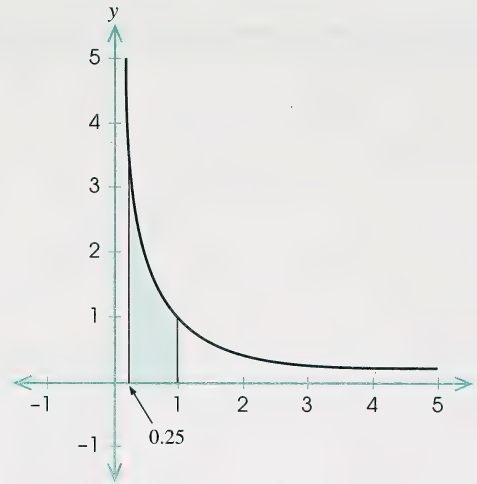
Shift X,θ,T 1 ÷ X,θ,T Shift → 1 Shift → 4 Shift → 9) EXE

The area is approximately 1.386 294 361.

Check

$\ln 4 = 1.386\ 294\ 361$

c.



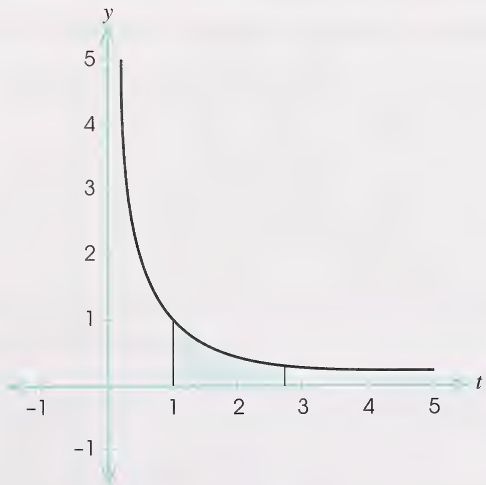
Shift X,θ,T 1 ÷ X,θ,T Shift → 1 Shift → 0 • 2 5 Shift → 9) EXE

The area is approximately $-1.386\ 294\ 361$.

Check

$\ln 0.25 = -1.386\ 294\ 361$

d.



Shift | X,θ,T | 1 | ÷ | X,θ,T | Shift | → | 1 | Shift | →

2 | • | 7 | 1 | 8 | 2 | 8 | 1 | 8 | 2 | 8 | 4 | 5

9 | Shift | → | 9 |) | EXE

The area is approximately 0.999 999 9998.

Check

$\ln 2.718\,281\,828\,459 = 0.999\,999\,9998$

2. a. $y = x^2 \ln x$

$$\begin{aligned}\frac{dy}{dx} &= x^2 \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}\left(x^2\right)\\ &= x^2 \bullet \frac{1}{x} + (\ln x)(2x)\\ &= x + 2x\ln x\end{aligned}$$

b. $y = 3\ln x^4$

$$\begin{aligned}\frac{dy}{dx} &= 3 \bullet \frac{1}{x^4} \bullet 4x^3\\ &= \frac{12}{x}\end{aligned}$$

c. $y = \ln(\sin x)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sin x} \bullet \frac{d}{dx}(\sin x)\\ &= \frac{\cos x}{\sin x}\\ &= \cot x\end{aligned}$$

d. $y=\ln\dfrac{2x-1}{x-1}$

$$\frac{dy}{dx}=\frac{1}{\frac{2x-1}{x-1}}\bullet\frac{d}{dx}\left(\frac{2x-1}{x-1}\right)$$
$$=\frac{x-1}{2x-1}\left[\frac{(x-1)\frac{d}{dx}(2x-1)-(2x-1)\frac{d}{dx}(x-1)}{(x-1)^2}\right]$$

quotient rule

$$=\frac{x-1}{2x-1}\left[\frac{(x-1)(2)-(2x-1)(1)}{(x-1)^2}\right]$$
$$=\frac{x-1}{2x-1}\left[\frac{2x-2-2x+1}{(x-1)^2}\right]$$
$$=\frac{-1(x-1)}{(2x-1)(x-1)^2}$$
$$=\frac{-1}{(2x-1)(x-1)}$$

3. a. $y=x^5\ln x^5$

$$\frac{dy}{dx}=x^5\frac{d}{dx}\left(\ln x^5\right)+\left(\ln x^5\right)\frac{d}{dx}\left(x^5\right)$$
$$=x^5\bullet\frac{1}{x^5}\bullet 5x^4+\left(\ln x^5\right)\left(5x^4\right)$$
$$=5x^4+5x^4\ln x^5$$
$$=5x^4\left(1+\ln x^5\right)$$

b. $y=x(\ln x)^2$

$$\frac{dy}{dx}=x\frac{d}{dx}(\ln x)^2+(\ln x)^2\frac{d}{dx}(x)$$
$$=x\bullet 2(\ln x)^1\left(\frac{1}{x}\right)+(\ln x)^2(1)$$
$$=2\ln x+(\ln x)^2$$
$$=(\ln x)(2+\ln x)$$

c. $y=\ln(\ln x)$

$$\frac{dy}{dx}=\frac{1}{\ln x}\bullet\frac{d}{dx}(\ln x)$$
$$=\frac{1}{\ln x}\bullet\frac{1}{x}$$
$$=\frac{1}{x\ln x}$$

d. $y=\ln[\sec(\ln x)]$

$$\frac{dy}{dx}=\frac{1}{\sec(\ln x)}\bullet\frac{d}{dx}[\sec(\ln x)]$$
$$=\frac{1}{\cancel{\sec(\ln x)}}\bullet\cancel{\sec(\ln x)}\bullet\tan(\ln x)\bullet\frac{d}{dx}(\ln x)$$
$$=[\tan(\ln x)]\bullet\frac{1}{x}$$
$$=\frac{\tan(\ln x)}{x}$$

4. **a.** $\int \frac{2}{x}\,dx = 2\int \frac{dx}{x}$

$$= 2\ln|x| + C$$

b. Assume $F(x) = 4a\ln|x-1|$.

$$\frac{d}{dx}F(x) = \frac{4a}{x-1}(1)$$
$$= \frac{4a}{x-1}$$

Equate to the integral, and solve for a.

$$\frac{4a}{x-1} = \frac{4}{x-1}$$
$$a = 1$$

$$\therefore \int \frac{4}{x-1}\,dx = 4\ln|x-1| + C$$

c. Assume $F(x) = a\ln\left|1-x^2\right|$.

$$\frac{d}{dx}F(x) = \frac{a}{1-x^2}(-2x)$$
$$= -\frac{2ax}{1-x^2}$$

Solve for a.

$$-\frac{2ax}{1-x^2} = \frac{x}{1-x^2}$$
$$-2a = 1$$
$$a = -\frac{1}{2}$$

$$\therefore \int \frac{x\,dx}{1-x^2} = -\frac{1}{2}\ln\left|1-x^2\right| + C$$

d. Assume $F(x) = \frac{a(\ln x)^4}{4}$.

$$\frac{d}{dx}F(x) = \frac{4a(\ln x)^3}{4} \bullet \frac{d(\ln x)}{dx}$$
$$= a(\ln x)^3 \bullet \frac{1}{x}$$
$$= \frac{a(\ln x)^3}{x}$$

Solve for a.

$$\frac{a(\ln x)^3}{x} = \frac{(\ln x)^3}{x}$$
$$a = 1$$

$$\therefore \int \frac{(\ln x)^3}{x}\,dx = \frac{(\ln x)^4}{4} + C$$

e. Assume $F(x) = a\ln\left|x^3+1\right|$.

$$\frac{d}{dx}F(x) = \frac{a}{x^3+1}\bullet\left(3x^2\right)$$
$$= \frac{3ax^2}{x^3+1}$$

Solve for a.

$$\frac{3ax^2}{x^3+1} = \frac{x^2}{x^3+1}$$
$$3a = 1$$
$$a = \frac{1}{3}$$

$$\therefore \int \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\ln\left|x^3+1\right| + C$$

f. Assume $F(x) = a\ln|2+\sin x|$.

$$\frac{d}{dx}F(x) = \frac{a}{2+\sin x}(\cos x)$$
$$= \frac{a\cos x}{2+\sin x}$$

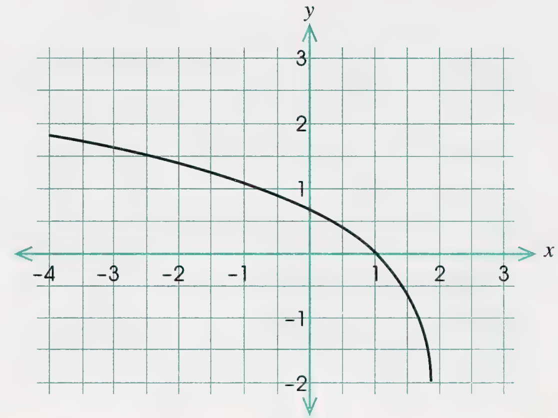
Solve for a.

$$\frac{a\cos x}{2+\sin x} = \frac{\cos x}{2+\sin x}$$
$$a = 1$$

$$\therefore \int \frac{\cos x}{2+\sin x}\,dx = \ln|2+\sin x| + C$$
$$= \ln(2+\sin x) + C$$

Section 1: Activity 2

1. a.

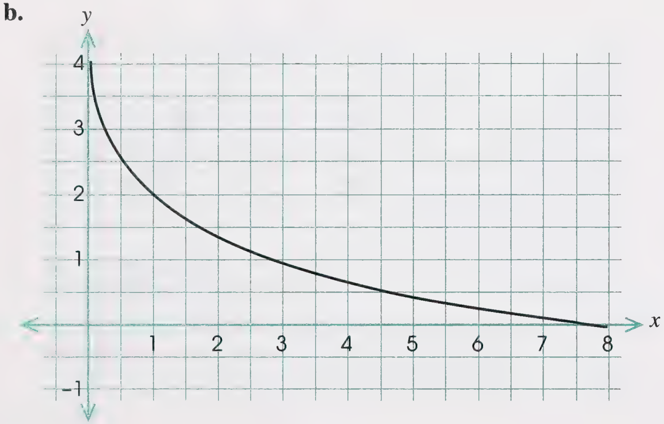


$$\begin{aligned} 2 - x &> 0 \\ -x &> -2 \\ x &< 2 \end{aligned}$$

The domain of $y = \ln(2 - x)$ is $(-\infty, 2)$, and the range is the set of reals.

The graph of $y = \ln(2 - x)$ is the graph of $y = \ln x$ translated 2 units horizontally; then reflected across $x = 2$.

b.



Since $\ln x$ is defined only when $x > 0$, the domain of $y = -\ln x + 2$ is $(0, \infty)$. The range is the set of reals.

The graph of $y = -\ln x + 2$ is the graph of $y = \ln x$ reflected across the x-axis and translated 2 units vertically upward.

2. a. Compare the derivatives of $y = \ln x$ and $y = \ln \frac{x}{a}$.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}\left(\ln \frac{x}{a}\right) = \frac{1}{\frac{x}{a}}\left(\frac{1}{a}\right) = \frac{1}{x}$$

Since the derivatives of the two functions are equal, they differ by a constant term at most. That is, $\ln \frac{x}{a} = \ln x + C$.

To obtain that constant, use the fact that $\ln 1 = 0$. Replace x by a.

$$\begin{aligned} \ln \frac{a}{a} &= \ln a + C \\ \ln 1 &= \ln a + C \\ 0 &= \ln a + C \\ C &= -\ln a \end{aligned}$$

$$\therefore \ \ln \frac{x}{a} = \ln x - \ln a$$

b. Compare the derivatives of $y = n\ln x$ and $y = \ln x^n$.

$$\frac{d}{dx}(n\ln x) = \frac{n}{x}$$

$$\frac{d}{dx}\left(\ln x^n\right) = \frac{1}{x^n}\left(nx^{n-1}\right) = \frac{n}{x}$$

Since the derivative of the two function are equal, they differ by a constant term at most. That is,

$\ln x^n = n\ln x + C$. To obtain that constant, use the fact that $\ln 1 = 0$. Replace x by 1.

$$\begin{aligned} \ln x^n &= n\ln x + C \\ \ln\left(1^n\right) &= n\ln 1 + C \\ 0 &= 0 + C \\ C &= 0 \end{aligned}$$

$$\therefore\ \ln x^n = n\ln x$$

3. a. $\ln 2 + \ln 3 = \ln 2(3)$
$= \ln 6$

Verify

LS	RS
$\ln 2 + \ln 3$	$\ln 6$
$= 0.693\,147 + 1.098\,612$	$= 1.791\,759$
$= 1.791\,759$	

LS = RS

b. $\ln 12 - \ln 2 = \ln\frac{12}{2}$
$= \ln 6$

Verify

LS	RS
$\ln 12 - \ln 2$	$\ln 6$
$= 2.484\,907 - 0.693\,147$	$= 1.791\,759$
$= 1.791\,759$	

LS = RS

c. $2\ln 4 = \ln 4^2$
$= \ln 16$

Verify

LS	RS
$2\ln 4$	$\ln 16$
$= 2(1.386\,294)$	$= 2.772\,589$
$= 2.772\,588$	

LS = RS

d. $\frac{1}{2}\ln 49 = \ln\sqrt{49}$

$= \ln 7$

LS	RS
$\frac{1}{2}\ln 49$	$\ln 7$
$=(0.5)(3.891\,820)$	$=1.945\,910$
$=1.945\,910$	

LS = RS

4. a. $\ln 6 = \ln(2)(3)$

$= \ln 2 + \ln 3$

$= a + b$

b. $\ln 12 = \ln 2^2(3)$

$= \ln 2^2 + \ln 3$

$= 2\ln 2 + \ln 3$

$= 2a + b$

c. $\ln\sqrt[3]{24} = \frac{1}{3}\ln(24)$

$= \frac{1}{3}\ln\left[(2)^3(3)\right]$

$= \frac{1}{3}(3\ln 2 + \ln 3)$

$= a + \frac{1}{3}b$

d. $\ln 1.5 = \ln\frac{3}{2}$

$= \ln 3 - \ln 2$

$= a - b$

5. a. $4\ln a + \ln b - \ln c = \ln\frac{a^4 b}{c}$

b. $\frac{1}{3}\ln a + \frac{1}{2}\ln b = \ln\sqrt[3]{a}\sqrt{b}$

c. $-\ln a = 0 - \ln a$

$= \ln 1 - \ln a$

$= \ln\frac{1}{a}$

6. $y = \ln(x+1) + \ln(x-1)$

$= \ln(x+1)(x-1)$

$= \ln\left(x^2 - 1\right)$

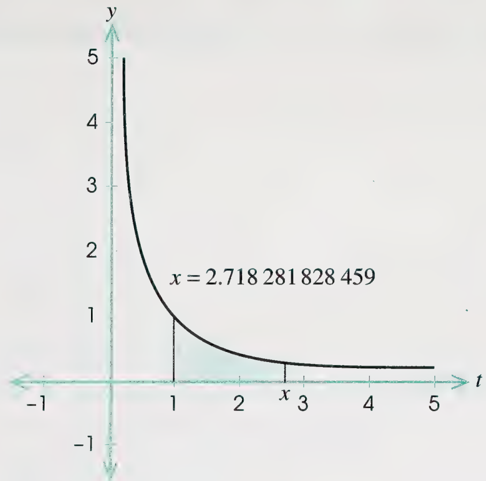
7.

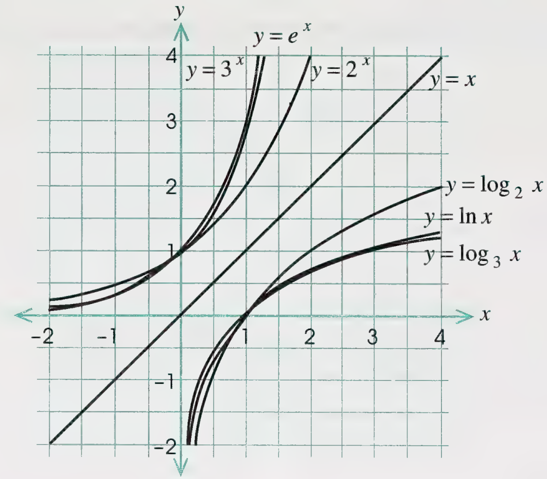
LS	RS
$\ln\left(x^2 - x\right)$	$\ln x + \ln(x-1)$
	$= \ln x(x-1)$
	$= \ln\left(x^2 - x\right)$

LS = RS

Section 1: Activity 3

1. The solution to $\ln x = 1$ is the value of x for which the area bounded above by $y = \frac{1}{t}$, and below by the x-axis over the interval $1, x$ is exactly 1. It is the shaded area in the following diagram. The corresponding x-value is approximately 2.718 281 828 459.





2.

Function	Inverse
$y = 2^x$	$x = 2^y$ or $y = \log_2 x$
$y = 3^x$	$x = 3^y$ or $y = \log_3 x$
$y = e^x$	$x = e^y$ or $y = \ln x$

3. **a.** base e **b.** base 10 **c.** base 2

4.

Function	Domain	Range
$y = e^x$	reals	$y > 0$
$y = \ln x$	$x > 0$	reals

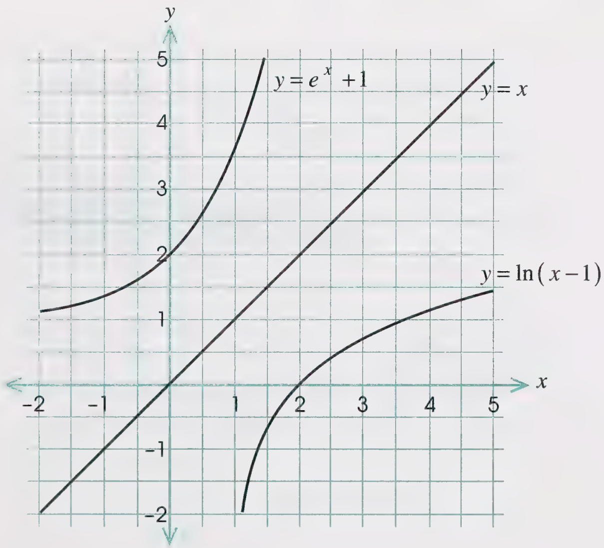
The domain and range of $y = e^x$ are the inverse of the domain and range of $y = \ln x$.

5. Since $e^x > 0$ for all x, there is no x-intercept.

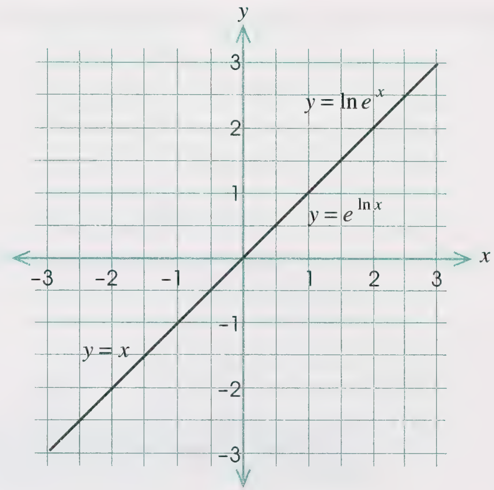
Since $e^0 = 1$, the y-intercept is 1.

6. The inverse is $x = \ln(y-1)$.

$$\therefore\ y - 1 = e^x$$
$$y = e^x + 1$$



7. a.



Both $y = \ln e^x$ and $y = e^{\ln x}$ are the same as $y = x$.

$$\begin{aligned} y &= \ln e^x \\ &= x \ln^e \\ &= x(1) \\ &= x \end{aligned} \qquad\qquad \begin{aligned} y &= e^{\ln x} \\ \ln y &= \ln e^{\ln x} \\ \ln y &= (\ln x)(\ln e) \\ \ln y &= \ln x \\ y &= x \end{aligned}$$

Both results are consistent with the fact that $y = e^x$ and $y = \ln x$ are inverse functions.

b. $$\begin{aligned} e^{\ln 3} &= e^{1.098\,612\,289\ldots} \\ &= 3 \end{aligned}$$

Section 1: Activity 4

1. **a.** $y = 2e^{3x}$

$$\frac{dy}{dx} = 2e^{3x}\frac{d}{dx}(3x)$$
$$= 2e^{3x}(3)$$
$$= 6e^{3x}$$

b. $y = e^{x}\ln x$

Apply the product rule.

$$\frac{dy}{dx} = e^{x}\frac{d}{dx}[\ln(x)] + \ln x\frac{d}{dx}\left(e^{x}\right)$$
$$= e^{x}\bullet\frac{1}{x} + e^{x}\ln x$$
$$= e^{x}\left(\frac{1}{x} + \ln x\right)$$

c. $y = xe^{x}$

$$\frac{dy}{dx} = x\frac{d}{dx}\left(e^{x}\right) + e^{x}\frac{d}{dx}(x)$$
$$= xe^{x} + e^{x}$$
$$= e^{x}(x+1)$$

2. **a.** $y = e^{x^2}$

$$\frac{dy}{dx} = e^{x^2}\frac{d}{dx}\left(x^2\right)$$
$$= 2xe^{x^2}$$

b. $y = \sin e^{x}$

$$= \left(\cos e^{x}\right)\frac{d}{dx}\left(e^{x}\right)$$
$$= e^{x}\cos e^{x}$$

c. $y = e^{-x} + e^{x}$

$$\frac{dy}{dx} = e^{-x}\frac{d}{dx}(-x) + e^{x}$$
$$= -e^{-x} + e^{x}$$

3. **a.** $\int e^{-x}\,dx = -e^{-x} + C$

b. $\int \frac{e^{\ln x}}{x}\,dx = ?$

Assume the primitive is $e^{\ln x}$.

$$\frac{d}{dx}F(x) = \frac{e^{\ln x}}{x}$$
$$\int \frac{e^{\ln x}}{x}\,dx = e^{\ln x} + C$$

c. $\int 3\sec^2 5x e^{\tan 5x}\, dx = ?$

Assume the primitive is $F(x) = 3ae^{\tan 5x} + C$.

$$\begin{aligned}\frac{d}{dx}F(x) &= 3ae^{\tan 5x}\frac{d}{dx}(\tan 5e)\\ &= 3ae^{\tan 5x}\left(\sec^2 5x\right)(5)\\ &= 15a\sec^2 5xe^{\tan 5x}\end{aligned}$$

Solve for a.

$$\begin{aligned}15a\sec^2 5xe^{\tan 5x} &= 3\sec^2 5xe^{\tan 5x}\\ 15a &= 3\\ a &= \frac{1}{5}\end{aligned}$$

$$\therefore \int 3\sec^2 5xe^{\tan 5x}\, dx = \frac{3}{5}e^{\tan 5x} + C$$

d. $\int e^{2x}\sin e^{2x}\, dx = ?$

Assume $F(x) = -a\cos e^{2x}$.

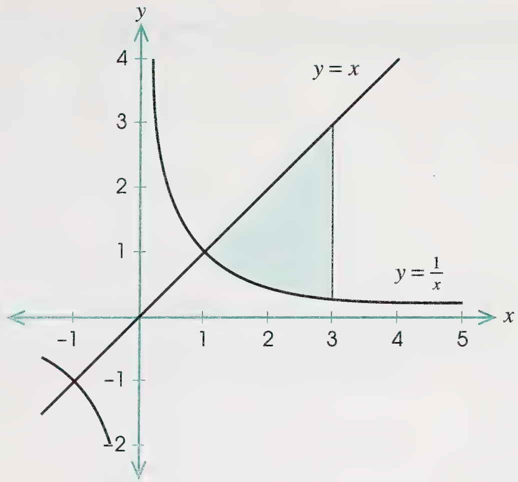
$$\begin{aligned}\frac{d}{dx}F(x) &= -a\left(-\sin e^{2x}\right)\left(e^{2x}\right)(2)\\ &= +2ae^{2x}\sin e^{2x}\end{aligned}$$

Solve for a.

$$\begin{aligned}2ae^{2x}\sin e^{2x} &= e^{2x}\sin e^{2x}\\ 2a &= 1\\ a &= \frac{1}{2}\end{aligned}$$

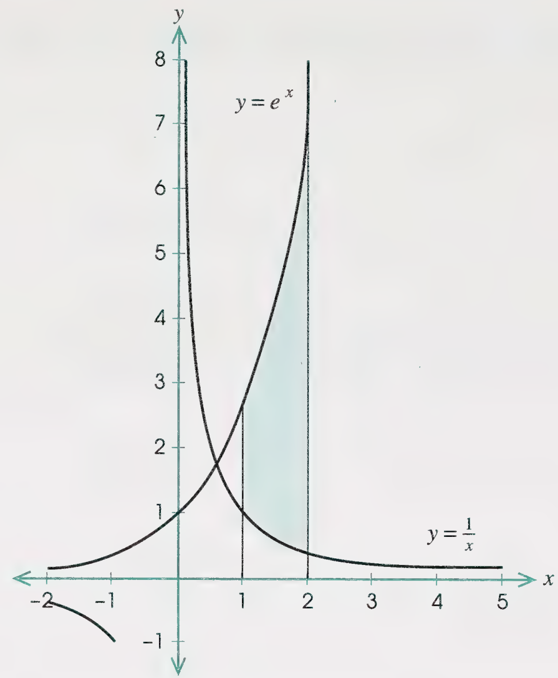
$$\therefore \int e^{2x}\sin e^{2x}\, dx = -\frac{1}{2}\cos e^{2x} + C$$

4.



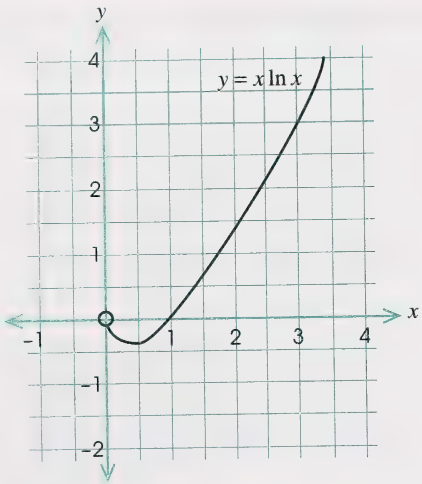
$$
{}_1A_3 = \int_1^3 [f(x) - g(x)]\, dx \text{, where } f(x) = x \text{ and } g(x) = \frac{1}{x}
$$
$$
= \int_1^3 \left[x - \frac{1}{x}\right] dx
$$
$$
= \left[\frac{x^2}{2} - \ln x\right]_1^3
$$
$$
= \left[\frac{3^3}{2} - \ln 3\right] - \left[\frac{1^2}{2} - \ln 1\right]
$$
$$
= \frac{9}{2} - \ln 3 - \frac{1}{2} + 0
$$
$$
= 4 - \ln 3
$$
$$
\doteq 2.901\,388
$$

5.



$$
{}_1A_2 = \int_1^2 [f(x) - g(x)]\, dx \text{, where } f(x) = e^x \text{ and } g(x) = \frac{1}{x}
$$
$$
= \int_1^2 \left[e^x - \frac{1}{x}\right] dx
$$
$$
= \left[e^x - \ln x\right]_1^2
$$
$$
= \left[e^2 - \ln 2\right] - \left[e^1 - \ln 1\right]
$$
$$
= e^2 - \ln 2 - e + 0
$$
$$
\doteq 3.977\,627\ldots
$$

6. a.



- Domain: Since $\ln x$ is defined only when $x > 0$, the domain of $y = x\ln x$ is $(0, \infty)$.
- Asymptotes and symmetry: none
- Intervals of increase and decrease

$$y = x\ln x$$
$$\frac{dy}{dx} = x \bullet \frac{1}{x} + (\ln x)(1)$$
$$= \ln x + 1$$

The function increases when $\frac{dy}{dx} > 0$.

$$\ln x + 1 > 0$$
$$\ln x > -1$$
$$x > e^{-1}$$
$$> \frac{1}{e}$$

The graph rises when $x \in \left(\frac{1}{e}, \infty\right)$.

The function decreases when $\frac{dy}{dx} < 0$.

$$\therefore \ \ln x + 1 < 0$$
$$x < \frac{1}{e}$$

The graph falls when $x \in \left(0, \frac{1}{e}\right)$.

- Maxima and minima: Since the graph falls to the right of $(0, 0)$, then $(0, 0)$, although not on the graph, defines a maximum point.

The minimum occurs at $x = \frac{1}{e}$.

When $x = \frac{1}{e}$, $y = \frac{1}{e}\ln\frac{1}{e}$

$$= \frac{1}{e}\ln e^{-1}$$

$$= \frac{1}{e}(-1)$$

$$= -\frac{1}{e}$$

$\left(\frac{1}{e}, -\frac{1}{e}\right)$ is an absolute minimum.

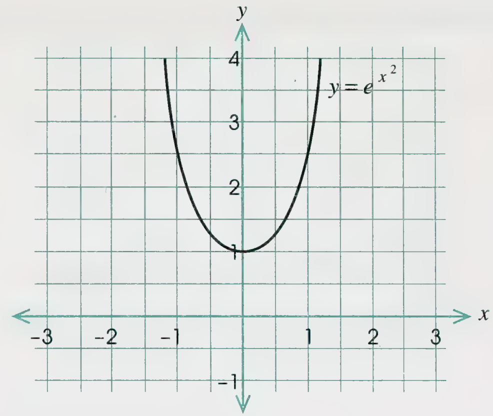
- Concavity: $\frac{dy}{dx} = \ln x + 1$

$$\frac{d^2y}{dx^2} = \frac{1}{x}$$

For $x > 0$, $\frac{d^2y}{dx^2} > 0$. The graph is concave upward throughout its domain.

- Points of inflection: none

b.



- Domain: reals

- Intercepts: When $x = 0$, $y = e^{0^2}$

$$= e^0$$

$$= 1$$

The graph crosses the y-axis at $(0, 1)$.
There is no x-intercept.

- Symmetry: The graph is symmetric with respect to the y-axis, as the original equation is unchanged when x is replaced by $-x$.

- Intervals of increase or decrease: Because $y = e^{x^2}$,

$$\frac{dy}{dx} = e^{x^2} \bullet \frac{d}{dx}\left(x^2\right)$$
$$= 2xe^{x^2}$$

The sign of the first derivative is the same as the sign of x.

$\frac{dy}{dx} > 0$ when $x > 0$, and $\frac{dy}{dx} < 0$ when $x < 0$

The graph rises when $x \in (0, \infty)$ and falls when $x \in (-\infty, 0)$. The turning point occurs when $x = 0$.

When $x = 0$, $y = 1$.

Therefore, $(0, 1)$ is a minimum point.

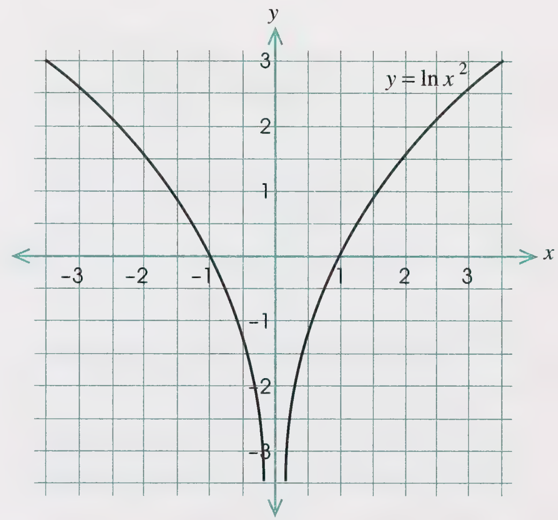
- Concavity: $\frac{dy}{dx} = 2xe^{x^2}$

$$\frac{d^2y}{dx^2} = 2x\frac{d}{dx}\left(e^{x^2}\right) + e^{x^2}\frac{d}{dx}(2x)$$
$$= 2x \bullet e^{x^2} \bullet 2x + e^{x^2} \bullet 2$$
$$= 4x^2e^{x^2} + 2e^{x^2}$$
$$= 2e^{x^2}\left(2x^2 + 1\right)$$

The second derivative is positive for all x; the graph is concave upward.

- Points of inflection: none

c.



- Domain: $x \neq 0$; the graph does not intersect the y-axis.
- Intercepts: The graph crosses the x-axis at $(\pm 1, 0)$.
- Asymptotes: Let $t = x^2$. Therefore, $y = \ln x^2 = \ln t$.

As $x \to 0^+$, $t \to 0^+$, and $\lim\limits_{x \to 0^+} \ln x^2 = \lim\limits_{t \to 0^+} \ln t$

$$= -\infty$$

As $x \to 0^-$, $t \to 0^+$, and $\lim\limits_{x \to 0^-} \ln x^2 = \lim\limits_{t \to 0^+} \ln t$

$$= -\infty$$

Therefore, the y-axis is a vertical asymptote.

- Symmetry: Since $\ln(-x)^2 = \ln x^2$, the curve is symmetric with respect to the y-axis.

- Intervals of increase and decrease:

$$y = \ln x^2$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x^2} \bullet \frac{d}{dx}\left(x^2\right) \\ &= \frac{2x}{x^2} \\ &= \frac{2}{x} \end{aligned}$$

When $x > 0$, $\frac{dy}{dx} > 0$ and the graph rises.

When $x < 0$, $\frac{dy}{dx} < 0$ and the graph falls.

- Concavity: $\frac{dy}{dx} = \frac{2}{x}$

$$\frac{d^2y}{dx^2} = -\frac{2}{x^2}$$

The second derivative is negative except at $x = 0$.

The graph is concave downward on both sides of the y-axis.

- Points of inflection: none

Section 1: Activity 5

1. a. $e^4 = \lim\limits_{n \to \infty} \left[1 + \frac{4}{n}\right]^n$

When $n = 100\,000\,000$, $\left[1 + \frac{4}{n}\right]^n \doteq 54.598\,145\,67$.

However, using a calculator, $e^4 \doteq 54.598\,150\,03$.

b. $e^{-2} = \lim\limits_{n \to \infty} \left[1 - \frac{2}{n}\right]^n$

When $n = 100\,000\,000$, $\left[1 - \frac{2}{n}\right]^n \doteq 0.135\,335\,280\,5$.

However, using a calculator, $e^{-2} \doteq 0.135\,335\,283$.

2. $P = \$10\,000$ $\quad i = \frac{0.09}{12} = 0.0075$ $\quad n = 12(4) = 48$

Compounding Interest Monthly

$$\begin{aligned} A &= 10\,000(1.0075)^{12(4)} \\ &= 14\,314.05 \end{aligned}$$

Therefore, the interest earned is
$(14\,314.05 - 10\,000 = 4314.05)$ \$4314.05.

Compounding Interest Continuously

$$\begin{aligned} A &= 10\,000\,e^{0.09(4)} \\ &= 14\,333.30 \end{aligned}$$

The interest earned with continuous compounding is \$4333.30.

You would receive $(4333.30 - 4314.05 = 19.25)$ \$19.25 of additional interest using continuous compounding .

3. $A = 3P$, $r =$ annual rate, and $t = 5$.

$$\begin{aligned} A &= Pe^{rt} \\ 3P &= Pe^{5r} \\ e^{5r} &= 3 \\ \ln e^{5r} &= \ln 3 \\ 5r &= \ln 3 \\ r &= \frac{\ln 3}{5} \\ &= 0.219\,722\,4577\ldots \\ &\doteq 22\% \end{aligned}$$

The interest rate should be approximately 22%.

4. a. Replace $\frac{3}{x}$ by t. If $t = \frac{3}{x}$, then as $x \to \infty$, $t \to 0$.

$$\begin{aligned} \lim_{x \to \infty} e^{\frac{3}{x}} &= \lim_{t \to 0} e^{t} \\ &= e^{0} \\ &= 1 \end{aligned}$$

b. Replace $\cos\left(x - \frac{\pi}{2}\right)$ by t. If $t = \cos\left(x - \frac{\pi}{2}\right)$, then $\left(x - \frac{\pi}{2}\right) \to \frac{\pi^{+}}{2}$ and $\cos\left(x - \frac{\pi}{2}\right) \to 0^{-}$ as $x \to \pi^{+}$.

$$\begin{aligned} \lim_{x \to \pi^{+}} e^{\cos\left(x - \frac{\pi}{2}\right)} &= \lim_{t \to 0^{-}} e^{t} \\ &= 1 \end{aligned}$$

c. Replace $\frac{x}{x+1}$ by t. As $x \to \infty$, $t \to 1$.

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(x)}{x+1} &= \lim_{t\to 1} \ln t\\ &= \ln 1\\ &= 0\end{aligned}$$

d. Let $t = \sin x$. As $x \to 0^-$, $\sin x \to 0^-$.

$$\begin{aligned}\lim_{x\to 0^-} \ln\left(e^{\sin x}\right) &= \lim_{t\to 0^-} \ln\left(e^{t}\right)\\ &= \ln\left(e^{0}\right)\\ &= \ln 1\\ &= 0\end{aligned}$$

e. Let $t = x+1$. As $x \to -1^+$, $t \to 0^+$.

$$\begin{aligned}\lim_{x\to -1^+} \ln(x+1) &= \lim_{t\to 0^+} \ln t\\ &= -\infty\end{aligned}$$

5. a. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$$\begin{aligned}d\left(e^x\right) &= \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{1}{2!}\bullet\frac{d}{dx}\left(x^2\right) + \frac{1}{3!}\bullet\frac{d}{dx}\left(x^3\right)\\ &\quad + \frac{1}{4!}\bullet\frac{d}{dx}\left(x^4\right) + \dots\\ &= 0 + 1 + \frac{2x}{2} + \frac{3x^2}{3(2!)} + \frac{4x^3}{4(3!)} + \dots\\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\\ &= e^x\end{aligned}$$

b.
$$\begin{aligned}e^1 &= 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!}\dots\\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}\dots\\ &\doteq 2.716\,666\end{aligned}$$

The actual value is $2.718\,281\,828\,459\,045\dots$.

Section 1: Activity 6

1. $2^7 = 128$ and $e^{7\ln 2} = 128$. The function $y = 2^x$ is the same as $y = e^{x\ln 2}$; the values are equal.

2.
$$\begin{aligned}4^5 &= e^{5\ln 4}\\ &\doteq e^{6.931\,471\,806}\end{aligned}$$

3. $y = 10^x$

$= e^{x \ln 10}$

4. $y = e^{x \ln 7}$

$= 7^x$

5. **a.** $e^{\ln 3^2} = 9 \quad \left(\text{since } e^{\ln x} = x\right)$

b. Since the exponential and logarithmic functions are inverses, $y = e^{\ln 2^x}$ is the same as $y = 2^x$.

6. $\frac{d}{dx}\left(e^x\right) = e^x \ln e$

$= e^x (1)$

$= e^x$

7. **a.** Apply the product rule.

$$\frac{dy}{dx} = 3^x \frac{d}{dx}(\sin x) + \sin x \bullet \frac{d}{dx}\left(3^x\right)$$

$$= 3^x \cos x + 3^x (\ln 3)(\sin x)$$

b. $\frac{dy}{dx} = 2^{x^2-6} (\ln 2) \bullet \frac{d}{dx}\left(x^2 - 6\right)$

$= 2x(\ln 2)\left(2^{x^2-6}\right)$

c. $\frac{dy}{dx} = 2^{\ln x} (\ln 2) \frac{d}{dx}(\ln x)$

$= \frac{(\ln 2) 2^{\ln x}}{x}$

d. $\frac{dy}{dx} = 10^{\tan 2x} (\ln 10) \frac{d}{dx}(\tan 2x)$

$= \left(2 \sec^2 2x\right)\left(10^{\tan 2x}\right)(\ln 10)$

e. $\frac{dy}{dx} = \frac{x\left(5^x\right)(\ln 5) - 5^x}{x^2}$

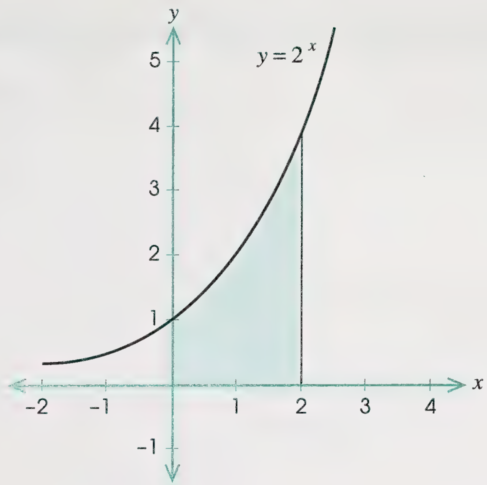
$= 5^x \frac{(x \ln 5 - 1)}{x^2}$

8. **a.** $\int 4^x \, dx = \frac{4^x}{\ln 4} + C$

b. $\int 3\left(2^x\right) dx = \frac{3\left(2^x\right)}{\ln 2} + C$

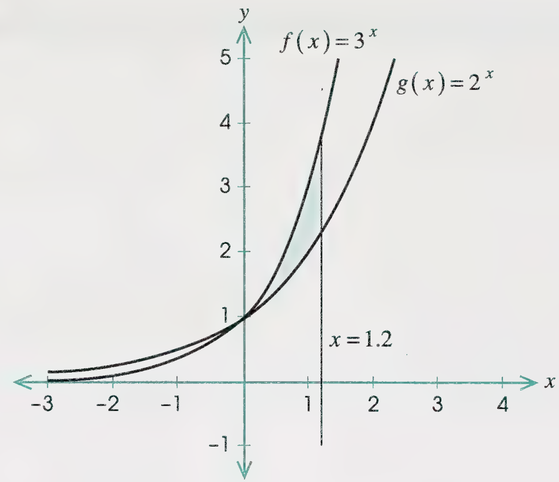
c. $\int \left(\sec^2 x\right) 10^{\tan x} \, dx = \frac{10^{\tan x}}{\ln 10} + C$

9.



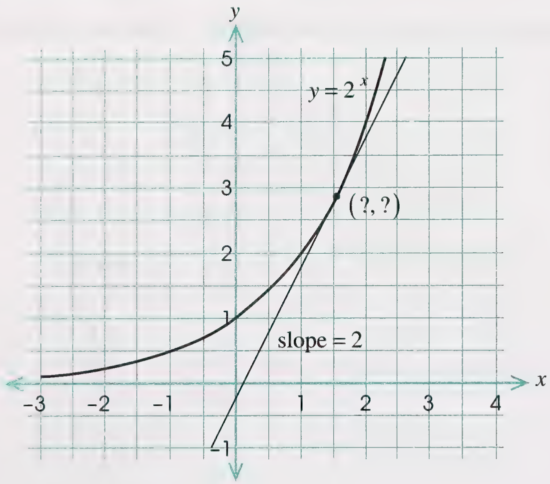
$$\begin{aligned}{}_0A_2 &= \int_0^2 2^x \, dx \\ &= \left[\frac{2^x}{\ln 2}\right]_0^2 \\ &= \frac{1}{\ln 2}\left[2^2 - 2^0\right] \\ &= \frac{3}{\ln 2} \\ &\doteq 4.328\end{aligned}$$

10.



$$\begin{aligned}{}_0A_{1.2} &= \int_0^{1.2} [f(x) - g(x)] \, dx \\ &= \int_0^{1.2} \left(3^x - 2^x\right) dx \\ &= \left[\frac{3^x}{\ln 3} - \frac{2^x}{\ln 2}\right]_0^{1.2} \\ &= \left[\frac{3^{1.2}}{\ln 3} - \frac{2^{1.2}}{\ln 2}\right] - \left[\frac{3^0}{\ln 3} - \frac{2^0}{\ln 2}\right] \\ &= \frac{3^{1.2}}{\ln 3} - \frac{1}{\ln 3} - \frac{2^{1.2}}{\ln 2} + \frac{1}{\ln 2} \\ &= \frac{1}{\ln 3}\left[3^{1.2} - 1\right] - \frac{1}{\ln 2}\left[2^{1.2} - 1\right] \\ &\doteq 0.620\end{aligned}$$

11.



$$y = 2^x$$

$$\frac{dy}{dx} = 2^x \ln 2$$

$$\therefore\ 2^x \ln 2 = 2$$

The slope at the required point is 2.

$$2^x = \frac{2}{\ln 2}$$

$$\ln 2^x = \ln\left[\frac{2}{\ln 2}\right]$$

$$x \ln 2 = \ln 2 - \ln(\ln 2)$$

$$x = \frac{\ln 2 - \ln(\ln 2)}{\ln 2}$$

$$\doteq 1.529$$

$$\therefore\ y \doteq 2.885$$

12. a. $\log_7 6 = \frac{\ln 6}{\ln 7}$

6 ln ÷ 7 ln =

0.920782221

b. $\log_4 0.2 = \frac{\ln 0.2}{\ln 4}$

0 • 2 ln ÷ 4 ln =

-1.160964047

13. a. $y = \log x$

$$= \log_{10} x$$

$$\frac{dy}{dx} = \frac{1}{x \ln 10}$$

b. $y = \log_8 (3x)$

$$\frac{dy}{dx} = \frac{1(3)}{3x \ln 8}$$

$$= \frac{1}{x \ln 8}$$

Section 1: Follow-up Activities

Extra Help

1. a. $y = \ln\frac{x-2}{x-3}$

$= \ln(x-2) - \ln(x-3)$

$\frac{dy}{dx} = \frac{1}{x-2} - \frac{1}{x-3}$

$= \frac{(x-3)-(x-2)}{(x-2)(x-3)}$

$= \frac{-1}{(x-2)(x-3)}$

b. $y = \ln\frac{\sqrt{x-1}}{x}$

$= \frac{1}{2}\ln(x-1) - \ln x$

$\frac{dy}{dx} = \frac{1}{2(x-1)} - \frac{1}{x}$

$= \frac{x-2(x-1)}{2x(x-1)}$

$= \frac{-x+2}{2x(x-1)}$

c. $y = \ln 3^{x-5}$

$= (x-5)\ln 3$

$= x\ln 3 - 5\ln 3$

$\frac{dy}{dx} = \ln 3$

d. $y = \ln 5(x-8)^{11}$

$= \ln 5 + 11\ln(x-8)$

$\frac{dy}{dx} = \frac{11}{x-8}$

2. $y = \log 3x$

$= \log 3 + \log x$

$\frac{dy}{dx} = 0 + \frac{1}{x\ln 10}$

$= \frac{1}{x\ln 10}$

Enrichment

1. a.

$$
\begin{aligned}
y &= x^{\cos x} \\
\ln y &= (\cos x)(\ln x) \\
\frac{1}{y} \bullet \frac{dy}{dx} &= \frac{\cos\ x}{x} + (\ln x)(-\sin\ x) \\
\frac{1}{y} \bullet \frac{dy}{dx} &= \frac{\cos\ x}{x} - (\ln x)(\sin\ x) \\
\frac{dy}{dx} &= y\left[\frac{\cos\ x}{x} - (\ln x)(\sin\ x)\right] \\
\frac{dy}{dx} &= x^{\cos x}\left[\frac{\cos x}{x} - (\ln x)(\sin x)\right]
\end{aligned}
$$

b.

$$
\begin{aligned}
y &= x^n \\
\ln y &= n \ln x \\
\frac{1}{y} \bullet \frac{dy}{dx} &= \frac{n}{x} \\
\frac{dy}{dx} &= \frac{ny}{x} \\
&= \frac{nx^n}{x} \\
&= nx^{n-1}
\end{aligned}
$$

c.

$$
\begin{aligned}
y &= x^x \\
\ln y &= x \ln x \\
\frac{1}{y} \bullet \frac{dy}{dx} &= \frac{x}{x} + (\ln x)(1) \\
\frac{1}{y} \bullet \frac{dy}{dx} &= 1 + \ln x \\
\frac{dy}{dx} &= y(1 + \ln x) \\
&= x^x (1 + \ln x)
\end{aligned}
$$

d.

$$
\begin{aligned}
y^2 &= x(x-1) \\
\ln y^2 &= \ln x(x-1) \\
2 \ln y &= \ln x + \ln(x-1) \\
\frac{2}{y} \bullet \frac{dy}{dx} &= \frac{1}{x} + \frac{1}{x-1} \\
\frac{2}{y} \bullet \frac{dy}{dx} &= \frac{2x-1}{x(x-1)} \\
\frac{dy}{dx} &= \frac{y(2x-1)}{2x(x-1)} \\
&= \pm \frac{[x(x-1)]^{\frac{1}{2}} (2x-1)}{2x(x-1)}
\end{aligned}
$$

e.

$$y=\frac{(2x-1)^{\frac{1}{3}}(x+1)^{\frac{1}{2}}}{x-1}$$

$$\ln y=\ln\frac{(2x-1)^{\frac{1}{3}}(x+1)^{\frac{1}{2}}}{x-1}$$

$$\ln y=\frac{1}{3}\ln(2x-1)+\frac{1}{2}\ln(x+1)-\ln(x+1)$$

$$\frac{1}{y}\bullet\frac{dy}{dx}=\frac{2}{3(2x-1)}+\frac{1}{2(x+1)}-\frac{1}{x+1}$$

$$\frac{dy}{dx}=y\left[\frac{2}{3(2x-1)}+\frac{1}{2(x+1)}-\frac{1}{x+1}\right]$$

2.

$$y^2=x(x-1)$$

$$y^2=x^2-x$$

$$2y\frac{dy}{dx}=2x-1$$

$$\frac{dy}{dx}=\frac{2x-1}{2y}$$

$$=\frac{2x-1}{2\left(\pm\sqrt{x(x-1)}\right)}$$

$$=\frac{2x-1}{2\left(\pm\sqrt{x(x-1)}\right)}\bullet\frac{\sqrt{x(x-1)}}{\sqrt{x(x-1)}}$$

$$=\pm\frac{[x(x-1)]^{\frac{1}{2}}(2x-1)}{2x(x-1)}$$

The answer is the same as the answer to question 1.d.

Section 2: Activity 1

1. The procedure for answering Example 2 is the same as finding compound interest on a deposit.

$A=P(1+i)^n$, where $i=0.025$ and $P=30\ 000\ 000$.

$\therefore\ A=30\ 000\ 000(1.025)^n$

You are required to find *n*, if $A=60\ 000\ 000$.

$$\begin{aligned} 60\ 000\ 000&=30\ 000\ 000(1.025)^n\\ 2&=(1.025)^n\\ \ln 2&=\ln(1.025)^n\\ \ln 2&=n\ln(1.025)\\ n&=\frac{\ln 2}{\ln 1.025}\\ &\doteq 28 \end{aligned}$$

Canada's population will double in about 28 years.

The two answers are the same since $A=30\ 000\ 000(1.025)^n$ can be transformed into $A=30\ 000\ 000e^{(\ln 1.025)t}$. Remember that $e^{x\ln b}=b^x$. Therefore, $(1.025)^n=e^{(\ln 1.025)t}$.

2. **a.** $y_0 = 2\,000\,000$

$\therefore\ y = 2\,000\,000\,e^{kt}$

To find k, use $y = 4\,000\,000$ at $t = 30$ d.

$$\begin{aligned} 4\,000\,000 &= 2\,000\,000\,e^{30k} \\ 2 &= e^{30k} \\ \ln 2 &= \ln e^{30k} \\ \ln 2 &= 30k \\ k &= \frac{\ln 2}{30} \end{aligned}$$

$\therefore\ y = 2\,000\,000\,e^{(\ln 2)\frac{t}{30}}$

After 90 d, $y = 2\,000\,000\,e^{(\ln 2)\frac{90}{30}}$

$\doteq 16\,000\,000$

There will be approximately 16 million dust mites.

b. $y = (2\,000\,000)2^{\frac{t}{30}}$

After 90 d, $y = (2\,000\,000)2^{\frac{90}{30}}$

$= (2\,000\,000)(8)$

$= 16\,000\,000$

There will be approximately 16 million dust mites.

c. These approaches are equivalent since $2^{\frac{t}{30}} = e^{(\ln 2)\frac{t}{30}}$.

3. Use $y = y_0 e^{kt}$.

To find k, use $y = 3y_0$ at $t = 60$ min.

$$\begin{aligned} \therefore\ 3y_0 &= y_0 e^{60k} \\ 3 &= e^{60k} \\ \ln 3 &= \ln e^{60k} \\ \ln 3 &= 60k \\ k &= \frac{\ln 3}{60} \end{aligned}$$

$\therefore y = y_0 e^{(\ln 3)\frac{t}{60}}$

When the population doubles, $y = 2y_0$.

$$\begin{aligned} \therefore\ 2y_0 &= y_0 e^{(\ln 3)\frac{t}{60}} \\ 2 &= e^{(\ln 3)\frac{t}{60}} \\ \ln 2 &= \frac{(\ln 3)}{60}t \\ t &= \frac{60(\ln 2)}{\ln 3} \\ &\doteq 38 \end{aligned}$$

The bacteria doubles in about 38 min.

4. $\frac{dy}{dt} \alpha y$

$\therefore \frac{dy}{dt} = ky$

$\therefore y = y_0 e^{kt}$

To find k, use $y = 2y_0$ at $t = 3$ a.

$$\begin{aligned} \therefore 2y_0 &= y_0 e^{2k} \\ 2 &= e^{2k} \\ \ln 2 &= \ln e^{2k} \\ \ln 2 &= 2k \\ \therefore k &= \frac{\ln 2}{2} \end{aligned}$$

5. $\frac{dy}{dt} \alpha y$

$\therefore \frac{dy}{dt} = ky$

$\therefore y = y_0 e^{kt}$

Now $y_0 = 200$.

$\therefore y = 200e^{kt}$

To find k, use $y = 300$ at $t = 1$ year.

$$\begin{aligned} \therefore 300 &= 200e^{1k} \\ 1.5 &= e^{k} \\ \ln 1.5 &= \ln e^{k} \\ \ln 1.5 &= k \end{aligned}$$

$\therefore y = 200e^{(\ln 1.5)t}$

When $y = 1000$, $1000 = 200e^{(\ln 1.5)t}$

$$\begin{aligned} 5 &= e^{(\ln 1.5)t} \\ \ln 5 &= (\ln 1.5)t \\ t &= \frac{\ln 5}{\ln 1.5} \\ &\doteq 4 \end{aligned}$$

The stamp will be worth \$1000 in about four years.

6. Begin by finding the growth function for the red ants.

$y_0 = 30\,000$

$\therefore y = 30\,000e^{kt}$

To find k, use $y = 60\,000$ at $t = 30$ d.

$$
\begin{aligned}
60\,000 &= 30\,000e^{30k} \\
2 &= e^{30k} \\
\ln 2 &= \ln e^{30k} \\
\ln 2 &= 30k \\
k &= \frac{\ln 2}{30}
\end{aligned}
$$

$$\therefore\ y = 30\,000e^{(\ln 2)\frac{t}{30}}$$

Next, find the growth function for the black ants.

$$
\begin{aligned}
y_0 &= 45\,000 \\
\therefore\ y &= 45\,000e^{kt}
\end{aligned}
$$

To find k, use $y = 90\,000$ at $t = 45$ d.

$$
\begin{aligned}
90\,000 &= 45\,000e^{45k} \\
2 &= e^{45k} \\
\ln 2 &= \ln e^{45k} \\
\ln 2 &= 45k \\
k &= \frac{\ln 2}{45}
\end{aligned}
$$

$$\therefore\ y = 45\,000e^{(\ln 2)\frac{t}{45}}$$

$$
\begin{aligned}
45\,000e^{(\ln 2)\frac{t}{45}} &= 30\,000e^{(\ln 2)\frac{t}{30}} \\
\frac{45\,000}{30\,000} &= \frac{e^{(\ln 2)\frac{t}{30}}}{e^{(\ln 2)\frac{t}{45}}} \\
1.5 &= e^{(\ln 2)\frac{t}{30}-(\ln 2)\frac{t}{45}} \\
1.5 &= e^{(\ln 2)t\left[\frac{1}{30}-\frac{1}{45}\right]} \\
1.5 &= e^{(\ln 2)t\left[\frac{1}{90}\right]} \\
\ln 1.5 &= \frac{(\ln 2)t}{90} \\
t &= \frac{90\ln 1.5}{\ln 2} \\
&\doteq 53
\end{aligned}
$$

The populations will be the same after approximately 53 d.

Section 2: Activity 2

1. Use the relationship $b^x = e^{x\ln b}$. Now $b = 2$ and $x = -\frac{t}{h}$.

$$
\begin{aligned}
\therefore\ y &= y_0 2^{-\frac{t}{h}} \\
&= y_0 e^{-(\ln 2)\frac{t}{h}}
\end{aligned}
$$

Note that $k = -\frac{\ln 2}{h}$.

2.
$$\begin{aligned} y &= y_0 e^{kt} \\ 0.5 y_0 &= y_0 e^{k(5700)} \\ 0.5 &= e^{k(5700)} \\ \ln 0.5 &= k(5700) \\ k &= \frac{\ln 0.5}{5700} \end{aligned}$$

$$\therefore\ y = y_0 e^{(\ln 0.5)\frac{t}{5700}}$$

Find the number of years that must elapse before $y = \frac{y_0}{100}$.

$$\begin{aligned} \frac{y_0}{100} &= y_0 e^{(\ln 0.5)\frac{t}{5700}} \\ 0.01 &= e^{(\ln 0.5)\frac{t}{5700}} \\ \ln 0.01 &= (\ln 0.5)\frac{t}{5700} \\ t &= \frac{5700 \ln 0.01}{\ln 0.5} \\ &\doteq 37\,900 \end{aligned}$$

The individual lived about 37 900 years ago.

3.
$$\begin{aligned} y &= y_0 e^{kt} \\ 0.80 y_0 &= y_0 e^{k(1)} \\ 0.80 &= e^{k} \\ \ln 0.80 &= k \\ k &= \ln 0.80 \end{aligned}$$

$$\therefore\ y = y_0 e^{(\ln 0.80)t}$$

Find the percentage that remains after three minutes.

$$\begin{aligned} y &= y_0 e^{(\ln 0.80)(3)} \\ &= 0.512 y_0 \end{aligned}$$

Approximately 51% of the original sample remains.

4. $\frac{dD}{dt} \propto D$

$$\begin{aligned} \frac{dD}{dt} &= kD \\ \frac{1}{D} \bullet \frac{dD}{dt} &= k \\ \int \frac{dD}{D} &= \int k\, dt \\ \ln D &= kt + C \end{aligned}$$

APPENDIX

Glossary

Suggested Answers

7. Represent the temperature difference by D.

$$\frac{dD}{dt} \propto D$$

$$\therefore \frac{dD}{dt} = kD$$

$$\frac{1}{D} \bullet \frac{dD}{dt} = k$$

$$\int \frac{dD}{D} = \int k\,dt$$

$$\ln D = kt + C$$

Evaluate C. The initial temperature difference was $90°\text{C} - (-40°\text{C}) = 130°\text{C}$.

$$\ln 130 = k(0) + C$$

$$C = \ln 130$$

$$\therefore \ln D = kt + \ln 130$$

$$\ln D - \ln 130 = kt$$

$$\ln \frac{D}{130} = kt$$

Evaluate k using a temperature difference of 90°C after three minutes.

$$\ln \frac{90}{130} = 3k$$

$$\therefore k = \frac{\ln\left(\frac{9}{13}\right)}{3}$$

$$\ln \frac{D}{130} = \frac{\left[\ln\left(\frac{9}{13}\right)\right]t}{3}$$

$$\frac{D}{130} = e^{\left[\ln\left(\frac{9}{13}\right)\right]\frac{t}{3}}$$

$$D = 130e^{\left[\ln\left(\frac{9}{13}\right)\right]\frac{t}{3}}$$

The pastries will reach 30°C when the temperature difference is 60°C.

$$\therefore 60 = 130e^{\left[\ln\left(\frac{9}{13}\right)\right]\frac{t}{3}}$$

$$\frac{6}{13} = e^{\left[\ln\left(\frac{9}{13}\right)\right]\frac{t}{3}}$$

$$\ln\left(\frac{6}{13}\right) = \frac{\left[\ln\left(\frac{9}{13}\right)\right]t}{3}$$

$$t = \frac{3\ln\left(\frac{6}{13}\right)}{\ln\left(\frac{9}{13}\right)}$$

$$\doteq 13$$

The pastries will reach 30°C after about thirteen minutes.

8. a. Let $y =$ value of the car at time t in years.

$$\therefore \frac{dy}{dt} \propto y \text{ or } \frac{dy}{dt} = kt$$

Therefore, $y = y_0 e^{kt}$, where $y_0 =$ original price.

$$\therefore y = 20\,000 e^{kt}$$

Substitute; $y = 3361.40$ when $t = 5$.

$$\begin{aligned} 3361.40 &= 20\,000 e^{5k} \\ \frac{3361.40}{20\,000} &= e^{5k} \\ \ln(0.168\,07) &= \ln e^{5k} \\ 5k &= \ln(0.168\,07) \\ k &= \frac{\ln(0.168\,07)}{5} \end{aligned}$$

$$\therefore y = 20\,000 e^{(\ln 0.168\,07)\frac{t}{5}}$$

To determine the half-life, the value of the car would be \$10 000.

$$\begin{aligned} 10\,000 &= 20\,000 e^{(\ln 0.168\,07)\frac{t}{5}} \\ 0.5 &= e^{(\ln 0.168\,07)\frac{t}{5}} \\ \ln 0.5 &= \frac{(\ln 0.168\,07)t}{5} \\ t &= \frac{5 \ln 0.5}{\ln(0.168\,07)} \\ &\doteq 2 \end{aligned}$$

The half-life of the car is about two years.

b. To determine the annual depreciation, determine the car's value after the first year.

$$\begin{aligned} \text{At } t = 1,\ y &= 20\,000 e^{(\ln 0.168\,07)\frac{1}{5}} \\ &= 14\,000 \end{aligned}$$

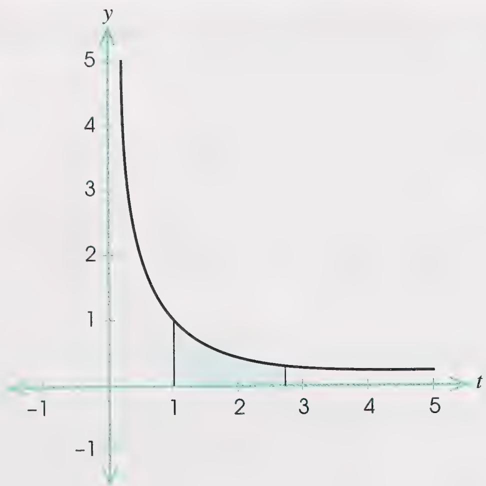
Therefore, the depreciation is \$6000 in the first year. The car depreciates at $\frac{6000}{20\,000} = 0.3$ or 30%.

The value of the car is 70% of its previous year's value.

Therefore, $y = (0.7)^t$, where $y =$ the car's value and $t =$ number of years.

9. Let y be the amount transmitted.
Let y_0 be the original amount.
Let x be the depth (in m).

d.



Shift X,θ,T 1 ÷ X,θ,T Shift → 1 Shift →

2 • 7 1 8 2 8 1 8 2 8 4 5

9 Shift → 9) EXE

The area is approximately 0.999 999 9998.

Check

$\ln 2.718\,281\,828\,459 = 0.999\,999\,9998$

2. **a.** $y = x^2 \ln x$

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}\left(x^2\right)$$
$$= x^2 \bullet \frac{1}{x} + (\ln x)(2x)$$
$$= x + 2x \ln x$$

b. $y = 3\ln x^4$

$$\frac{dy}{dx} = 3 \bullet \frac{1}{x^4} \bullet 4x^3$$
$$= \frac{12}{x}$$

c. $y = \ln(\sin x)$

$$\frac{dy}{dx} = \frac{1}{\sin x} \bullet \frac{d}{dx}(\sin x)$$
$$= \frac{\cos x}{\sin x}$$
$$= \cot x$$

d. $y = \ln \dfrac{2x-1}{x-1}$

$$\frac{dy}{dx} = \frac{1}{\frac{2x-1}{x-1}} \bullet \frac{d}{dx}\left(\frac{2x-1}{x-1}\right)$$

$$= \frac{x-1}{2x-1}\left[\frac{(x-1)\frac{d}{dx}(2x-1)-(2x-1)\frac{d}{dx}(x-1)}{(x-1)^2}\right]$$

quotient rule

$$= \frac{x-1}{2x-1}\left[\frac{(x-1)(2)-(2x-1)(1)}{(x-1)^2}\right]$$

$$= \frac{x-1}{2x-1}\left[\frac{2x-2-2x+1}{(x-1)^2}\right]$$

$$= \frac{-1(x-1)}{(2x-1)(x-1)^2}$$

$$= \frac{-1}{(2x-1)(x-1)}$$

3. a. $y = x^5 \ln x^5$

$$\frac{dy}{dx} = x^5 \frac{d}{dx}\left(\ln x^5\right) + \left(\ln x^5\right)\frac{d}{dx}\left(x^5\right)$$

$$= x^5 \bullet \frac{1}{x^5} \bullet 5x^4 + \left(\ln x^5\right)\left(5x^4\right)$$

$$= 5x^4 + 5x^4 \ln x^5$$

$$= 5x^4\left(1 + \ln x^5\right)$$

b. $y = x(\ln x)^2$

$$\frac{dy}{dx} = x\frac{d}{dx}(\ln x)^2 + (\ln x)^2 \frac{d}{dx}(x)$$

$$= x \bullet 2(\ln x)^1\left(\frac{1}{x}\right) + (\ln x)^2(1)$$

$$= 2\ln x + (\ln x)^2$$

$$= (\ln x)(2 + \ln x)$$

c. $y = \ln(\ln x)$

$$\frac{dy}{dx} = \frac{1}{\ln x} \bullet \frac{d}{dx}(\ln x)$$

$$= \frac{1}{\ln x} \bullet \frac{1}{x}$$

$$= \frac{1}{x \ln x}$$

d. $y = \ln[\sec(\ln x)]$

$$\frac{dy}{dx} = \frac{1}{\sec(\ln x)} \bullet \frac{d}{dx}[\sec(\ln x)]$$

$$= \frac{1}{\cancel{\sec(\ln x)}} \bullet \cancel{\sec(\ln x)} \bullet \tan(\ln x) \bullet \frac{d}{dx}(\ln x)$$

$$= [\tan(\ln x)] \bullet \frac{1}{x}$$

$$= \frac{\tan(\ln x)}{x}$$

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